Alternating conditional gradient method for convex feasibility problems *

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Abstract

The classical convex feasibility problem in a finite dimensional Euclidean space consists of finding a point in the intersection of two convex sets. In the present paper we are interested in two particular instances of this problem. First, we assume to know how to compute an exact projection onto one of the sets involved and the other set is compact such that the conditional gradient (CondG) method can be used for computing efficiently an inexact projection on it. Second, we assume that both sets involved are compact such that the CondG method can be used for computing efficiently inexact projections on them. We combine alternating projection method with CondG method to design a new method, which can be seen as an inexact feasible version of alternate projection method. The proposed method generates two different sequences belonging to each involved set, which converge to a point in the intersection of them whenever it is not empty. If the intersection is empty, then the sequences converge to points in the respective sets whose distance between them is equal to the distance between the sets in consideration. Numerical experiments are provided to illustrate the practical behavior of the method.

Keywords: Convex feasibility problem, alternating projection method, conditional gradient method, inexact projections.

AMS: 65K05, 90C30, 90C25.

1 Introduction

[9]

The *classic convex feasibility problem* consists of finding a point in the intersection of two sets. It is formally stated as follows:

$$\text{find } x_* \in A \cap B, \tag{1}$$

where $A, B \subset \mathbb{R}^n$ are convex, closed, and nonempty sets. Although we are not concerned with practical issues at this time, we emphasize that several practical applications appear modeled

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as Problem (1); see for example [15, 16, 28] and references therein. Among the methods to solve Problem (1), the *alternating projection method* is one of the most interesting and popular, with a long history dating back to J. von Neumann [36]. Since this seminal work, the alternating projection method has attracted the attention of the scientific community working on optimization, papers dealing with this method include [1, 8, 12]. Perhaps one of the factors that explains this interest is its simplicity and ease of implementation, making application to large-scale problems very attractive. Several variants of this method have arisen and several theoretical and practical issues related to it have been discovered over the years, resulting in a wide literature on the subject. For a historical perspective of this method; see, for exemple [3] and a complete annotated bibliography of books and review can be found in [11].

The aim of this paper is to present a new method to solve Problem 1. The proposed method is based on the alternating projection method. For designing the method, the *conditional gradient method (CondG method)* also known as *Frank-Wolfe algorithm* developed by Frank and Wolfe in 1956 [21] (see also [32]) is used as the subroutine to compute feasible inexact projections on the set in consideration, which will be named as *alternating conditional gradient (ACondG) method*. More specifically, the CondG method can be used to minimize the distance from a given point to a convex compact set. Since the CondG method is a feasible directions method, by introducing an error criterion, it returns a feasible inexact projection into the set in consideration. In this way, a relative error criterion is introduced appropriately, in order to be able to control the behaviour of the sequence generated by the ACondG method. We present two versions of the method. In the first version, we assume that we know how to compute an exact project onto one of the sets involved. Besides, we assume that the other set is compact with a special structure such that CondG method can be used for computing efficiently feasible inexact projections on it. In the second version, we assume that both sets are compact with special structures such that the CondG method can be used for computing feasible inexact projections on them. The ACondG method proposed, generates two sequences $(x^k)_{k\in\mathbb{N}} \subset A$ and $(y^k)_{k\in\mathbb{N}} \subset B$. The main results of this paper are as follows. If $A \cap B \neq \varnothing$, then the sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ converge to a point x^* belonging to $A \cap B$. If $A \cap B = \varnothing$, then the sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$, converge respectively to $x^* \in A$ and $y^* \in B$ satisfying $\|x^* - y^*\| = dist(A, B)$, where $dist(A, B)$ denotes the distance between the sets A and B.

From a practical point of view, methods that use inexact projections are particularly interesting. Indeed, one drawback of methods that use exact projections is the need to solve a quadratic problem at each stage, which may substantially increase the cost per iteration if the dimension of the space is large. In fact, it may not be justified to carry out exact projections when the iterates are *far* from the solution of the problem. For this reason, seeking to make the alternating projection method more efficient, we use the CondG method to compute feasible inexact projections rather than exact ones. It is noteworthy that the CondG method is easy to implement, has low computational cost per iteration, and readily exploits separability and sparsity, resulting in high computational performance in different classes of compact sets, see [20, 22, 25, 30, 32]. Therefore, we believe that all of these features accredit the method as being quite appropriate for our purpose, which has also been used for similar aims. Indeed, the most direct precursor is [31] that uses CondG method to compute Euclidean projections in an accelerated first-order method. Another work along a similar vein is [9] which uses a CondG method variant to compute scaled-projections (not necessarily Euclidean) in a projected-variable metric algorithm. For more applications of CondG method, see also [13, 23, 26, 34]. As aforementioned, the proposed method performs projections alternately on the sets A and B only approximately, but projections become increasingly accurate in relation to the progress of previous iterations. Therefore, the resulting method can be seen as an inexact version of the classical alternate projection method. It is also worth noting that others approximate projections have been widely used in the literature. For instance, approximate projection can be performed by projecting onto the hyperplane separating the set and the point to be projected, see [24], and for more examples, see [4, 10, 17, 18, 19, 29]. However, inexact projections obtained in this way are infeasible to the set to be projected, in contrast with feasible inexact projections propose here. In fact, our method ensures that all the iterates are feasible, in that each one always belongs to one of the sets, which in itself might also be very attractive. In addition, for feasible regions that can only be accessed efficiently through a linear programming oracle our method would be of considerable interest. Furthermore, as far as we know, the combination of the conditional method with the alternate directions method for designing a new method to solve Problem (1) has not yet been considered.

The organization of the paper is as follows. In section 2, we present some notation and basic results used in our presentation. In section 3 we describe the conditional gradient method and present some results related to it. In sections 4 and 5 we present, respectively, the first and second version of the inexact alternating projection method to solve Problem (1). Some numerical experiments are provided in section 6. We conclude the paper with some remarks in section 7.

Notation. We denote: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\langle \cdot, \cdot \rangle$ is the usual inner product, $\| \cdot \|$ is the Euclidean norm, and $[v]_i$ is the i-th component of the vector v.

2 Preliminaries

In this section, we present some preliminary results used throughout the paper. The *projection onto a closed convex set* $C \subset \mathbb{R}^n$ is the map $P_C : \mathbb{R}^n \to C$ defined by $P_C(v) := \arg\min_{z \in C} \|v - z\|$. In the next lemma we present some important properties of the projection mapping.

Lemma 1. *Let $C \subset \mathbb{R}^n$ be any nonempty closed and convex set and P_C the projection mapping onto C. For all $v \in \mathbb{R}^n$, the following properties hold:*

(i) $\langle v - P_C(v), z - P_C(v) \rangle \le 0$*, for all* $z \in C$*;*

(ii) $\|P_C(v) - z\|^2 \le \|v - z\|^2 - \|P_C(v) - v\|^2$*, for all* $z \in C$*;*

(iii) the projection mapping P_C is continuous.

Proof. The items (i) and (iii) are proved in [5, Proposition 3.10, Theorem 3.14]. For item (ii), combine $\|v - z\|^2 = \|P_C(v) - v\|^2 + \|P_C(v) - z\|^2 - 2\langle P_C(v) - v, P_C(v) - z \rangle$ with item (i). □

The next lemma plays an important role in the remainder of this paper, similar results can be found in [2]. For the sake of completeness, we decided to present the proof here. Before stating the lemma we need a definition. Let $C, D \subset \mathbb{R}^n$ be convex, closed, and nonempty sets. Define the *distance between the sets C and D* by $dist(C, D) := \inf\{\|v - w\| : \ v \in C, w \in D\}$.

Lemma 2. *Let C be a compact and convex set and D be a closed and convex set. Assume that the sequences $(v^k)_{k \in \mathbb{N}} \subset C$ and $(w^k)_{k \in \mathbb{N}} \subset D$ satisfy the following two conditions:*

(c1) $\lim_{k \to +\infty} \|v^{k+1} - v^k\| = 0$ *and* $\lim_{k \to +\infty} \|w^{k+1} - w^k\| = 0$*;*

(c2) $\lim_{k\to+\infty}\|w^{k+1}-P_D(v^k)\|=0$ *and* $\lim_{k\to+\infty}\|v^{k+1}-P_C(w^{k+1})\|=0$.

Then, each cluster point $\bar v$ *of* $(v^k)_{k\in\mathbb{N}}$ *is a fixed point of* P_CP_D*, i.e.,* $\bar v = P_CP_D(\bar v)$*. Moreover,* $\lim_{k\to\infty}\|v^k-w^k\| = dist(C,D)$ *and* $\lim_{k\to\infty}(v^k-w^k) = P_{C-D}(0)$*, where* $P_{C-D}(0) := \arg\min_{v\in C, w\in D}\|v-w\|$.

Proof. Since C is bounded and $(v^k)_{k\in\mathbb{N}}\subset C$, it follows that $(v^k)_{k\in\mathbb{N}}$ is bounded. Moreover, due to $\|v^k-w^k\|\le\|v^k-P_D(v^k)\|+\|P_D(v^k)-w^{k+1}\|+\|w^{k+1}-w^k\|$, combining the second equality in $(c1)$ with the first equality in $(c2)$ and item (iii) of Lemma 1, we conclude that $(w^k)_{k\in\mathbb{N}}$ is also bounded. Let $\bar v\in C$ be a cluster point of $(v^k)_{k\in\mathbb{N}}$ and $(v^{k_j})_{j\in\mathbb{N}}$ be a subsequence of $(v^k)_{k\in\mathbb{N}}$ with $\lim_{j\to+\infty}v^{k_j}=\bar v$. Take $(w^{k_j+1})_{j\in\mathbb{N}}$ a subsequence of $(w^{k+1})_{k\in\mathbb{N}}$. Since $(w^{k+1})_{k\in\mathbb{N}}$ is bounded, there exists a cluster point $\bar w$ of $(w^{k_j+1})_{j\in\mathbb{N}}$ and $(w^{k_\ell+1})_{\ell\in\mathbb{N}}$ a subsequence of $(w^{k_j+1})_{j\in\mathbb{N}}$ with $\lim_{\ell\to+\infty}w^{k_\ell+1}=\bar w$. Furthermore, the corresponding subsequence $(v^{k_\ell})_{\ell\in\mathbb{N}}$ of $(v^{k_j})_{j\in\mathbb{N}}$ also satisfies $\lim_{\ell\to+\infty}v^{k_\ell}=\bar v$. Due to $(\|v^{k+1}-v^k\|^2)_{k\in\mathbb{N}}$ and $(\|w^{k+1}-w^k\|^2)_{k\in\mathbb{N}}$ converge to zero, we also have $\lim_{\ell\to+\infty}v^{k_\ell+1}=\bar v$ and $\lim_{\ell\to+\infty}w^{k_\ell}=\bar w$. Moreover, it follows from Lemma 1 (ii) and conditions (c1) and (c2) that $\bar w = P_D(\bar v)$ and $\bar v = P_C(\bar w)$, respectively. Hence, the last equalities imply $\bar v = P_CP_D(\bar v)$ and we have $\|\bar v - P_D(\bar v)\| = dist(C,D)$, see [12, Theorem 2]. Therefore, we conclude that each cluster point $\bar v$ of $(v^k)_{k\in\mathbb{N}}$ is a fixed point of P_CP_D, i.e., $\bar v = P_CP_D(\bar v)$, which proves the first statement. Moreover, for each $\bar v\in C$ a cluster point of $(v^k)_{k\in\mathbb{N}}$, there exists a subsequence $(v^{k_\ell})_{\ell\in\mathbb{N}}$ of $(v^k)_{k\in\mathbb{N}}$ such that $\lim_{\ell\to+\infty}v^{k_\ell}=\bar v$ and a corresponding subsequence $(w^{k_\ell})_{\ell\in\mathbb{N}}$ of $(w^k)_{k\in\mathbb{N}}$ with $\lim_{\ell\to+\infty}w^{k_\ell}=P_D(\bar v)$ and $\|\bar v-P_D(\bar v)\| = dist(C,D)$. We claim that $(\|v^k-w^k\|)_{k\in\mathbb{N}}$ converges to the distance between C and D, i.e., $\lim_{k\to\infty}\|v^k-w^k\| = dist(C,D)$. Indeed, due to $(v^k)_{k\in\mathbb{N}}$ and $(w^k)_{k\in\mathbb{N}}$ being bounded, the sequence $(\|v^k-w^k\|)_{k\in\mathbb{N}}$ is also bounded. Let $\bar\alpha\ge 0$ be a cluster point of the sequence $(\|v^k-w^k\|)_{k\in\mathbb{N}}$ and $(\|v^{k_j}-w^{k_j}\|)_{j\in\mathbb{N}}$ be a subsequence of $(\|v^k-w^k\|)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty}\|v^{k_j}-w^{k_j}\|=\bar\alpha$. Take $\bar v\in C$ a cluster point of $(v^{k_j})_{j\in\mathbb{N}}$ and subsequences $(v^{k_\ell})_{\ell\in\mathbb{N}}$ of $(v^{k_\ell})_{\ell\in\mathbb{N}}$ and $(w^{k_\ell})_{\ell\in\mathbb{N}}$ of $(w^{k_j})_{j\in\mathbb{N}}$ such that $\lim_{\ell\to+\infty}w^{k_\ell}=P_D(\bar v)$ and $\|\bar v-P_D(\bar v)\| = dist(C,D)$. Since $\lim_{j\to+\infty}\|v^{k_j}-w^{k_j}\|=\bar\alpha$, we conclude that $\bar\alpha = \lim_{\ell\to+\infty}\|v^{k_\ell}-w^{k_\ell}\| = dist(C,D)$. Thus, $(\|v^k-w^k\|)_{k\in\mathbb{N}}$ has $dist(C,D)$ as unique cluster point. Therefore, $\bar\alpha = \lim_{\ell\to+\infty}\|v^{k_\ell}-w^{k_\ell}\| = dist(C,D)$ and this proves the second statement. Considering that D is a closed convex set and C is a compact convex set, it follows that $C-D$ is also a closed and convex set, which implies $P_{C-D}(0)$ is a singleton. Therefore, we obtain that $(v^k-w^k)_{k\in\mathbb{N}}$ converges to $P_{A-B}(0)$, i.e., $\lim_{k\to\infty}(v^k-w^k) = P_{C-D}(0)$ (see, [2, Lemma 2.3]), concluding the proof of the lemma. $\square$

Definition 1. *Let* S *be a nonempty subset of* $\mathbb{R}^n$*. A sequence* $(v^k)_{k\in\mathbb{N}}\subset\mathbb{R}^n$ *is said to be quasi-Fejér convergent to* S*, if and only if, for all* $v\in S$ *there exists* $k_0\ge 0$ *and a summable sequence* $(\epsilon_k)_{k\in\mathbb{N}}$*, such that* $\|v^{k+1}-v\|^2\le\|v^k-v\|^2+\epsilon_k$ *for all* $k\ge k_0$*. If for all* $k\in\mathbb{N}$*,* $\epsilon_k=0$*, the sequence* $(v^k)_{k\in\mathbb{N}}$ *is said to be Fejér convergent to* S*.*

In the following lemma, we state the main properties of quasi-Fejér sequences that we will need; a comprehensive study on this topic can be found in [17].

Lemma 3. *Let* $(v^k)_{k\in\mathbb{N}}$ *be quasi-Fejér convergent to* S*. Then, the following conditions hold:*

(i) *the sequence* $(v^k)_{k\in\mathbb{N}}$ *is bounded;*

(ii) *for all* $v\in S$*, the sequence* $(\|v^k-v\|)_{k\in\mathbb{N}}$ *is convergent.*

(iii) *if a cluster point* $\bar v$ *of* $(v^k)_{k\in\mathbb{N}}$ *belongs to* S*, then* $(v^k)_{k\in\mathbb{N}}$ *converges to* $\bar v$*.*

3 Conditional gradient (CondG) method

In the following we review the classical *conditional gradient method (CondG$_C$)* to compute feasible inexact projections onto a compact convex set C and some results related to it. We also prove two important inequalities related to this method that will be useful to establish our main results. For presenting the method, we assume the existence of a linear optimization oracle (or simply LO oracle) capable of minimizing linear functions over the constraint set C. We formally state the CondG$_C$ method to calculate an inexact projection of $v \in \mathbb{R}^n$ onto C, with the following input data: *a relative error tolerance function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following inequality*

$$\varphi_{\gamma,\theta,\lambda}(u, v, w) \leq \gamma\|v - u\|^2 + \theta\|w - v\|^2 + \lambda\|w - u\|^2, \qquad \forall\ u, v, w \in \mathbb{R}^n, \tag{2}$$

where $\gamma, \theta, \lambda \geq 0$ are given forcing parameters.

Algorithm 1: CondG$_C$ method w^+:=CondG$_C$ $(\varphi_{\gamma,\theta,\lambda}, u, v)$

Input: Take $\gamma, \theta, \lambda \in \mathbb{R}_+$, $v \in \mathbb{R}^n$, $u, w \in C$, and $\varphi_{\gamma,\theta,\lambda}$. Set $w_0 = w$ and $\ell = 0$.

Step 1. Use a LO oracle to compute an optimal solution z_ℓ and the optimal value s^*_ℓ as

$$z_\ell := \arg\min_{z\in C} \langle w_\ell - v,\ z - w_\ell\rangle, \qquad s^*_\ell := \langle w_\ell - v,\ z_\ell - w_\ell\rangle. \tag{3}$$

Step 2. If $-s^*_\ell \leq \varphi_{\gamma,\theta,\lambda}(u, v, w_\ell)$, then **stop**. Set $w^+ := w_\ell$.

Step 3. Compute $\alpha_\ell \in (0, 1]$ and $w_{\ell+1}$ as

$$w_{\ell+1} := w_\ell + \alpha_\ell(z_\ell - w_\ell), \qquad \alpha_\ell := \min\left\{1, \frac{-s^*_\ell}{\|z_\ell - w_\ell\|^2}\right\}. \tag{4}$$

Step 4. Set $\ell \leftarrow \ell + 1$, and go to step **1**.

Output: $w^+ := w_\ell$.

Let us describe the main features of CondG$_C$ method; for further details, see, for example, [6, 30, 31]. Let $v \in \mathbb{R}^n$, $\psi_v : \mathbb{R}^n \to \mathbb{R}$ be defined by $\psi_v(z) := \|z - v\|^2/2$, and $C \subset \mathbb{R}^n$ a convex compact set. It is worth mentioning that the above *CondG$_C$ method* can be viewed as a specialized version of the classic conditional gradient method applied to the problem $\min_{z\in C} \psi_v(z)$. In this case, (3) is equivalent to $s^*_\ell := \min_{z\in C}\langle \psi'_v(w_\ell),\ z - w_\ell\rangle$. Since ψ is convex we have

$$\psi_v(z) \geq \psi_v(w_\ell) + \langle \psi'_v(w_\ell),\ z - w_\ell\rangle \geq \psi_v(w_\ell) + s^*_\ell, \qquad \forall\ z \in C.$$

Set $w_* := \arg\min_{z\in C} \psi_v(z)$ and $\psi^* := \min_{z\in C} \psi_v(z)$. Letting $z = x_*$ in the last inequality we obtain that $\psi_v(w_\ell) \geq \psi^* \geq \psi(w_\ell) + s^*_\ell$, which implies that $s^*_\ell \leq 0$. Thus, we conclude that

$$-s^*_\ell = \langle v - w_\ell,\ z_\ell - w_\ell\rangle \geq 0 \geq \langle v - w_*,\ z - w_*\rangle, \qquad \forall\ z \in C.$$

Therefore, we state the stopping criteria as $-s^*_\ell \leq \varphi_{\gamma,\theta,\lambda}(u, v, w_\ell)$. Moreover, if the *CondG$_C$* method computes $w_\ell \in C$ satisfying $-s^*_\ell \leq \varphi_{\gamma,\theta,\lambda}(u, v, w_\ell)$, then the method terminates. Otherwise, it computes the stepsize $\alpha_\ell = \arg\min_{\alpha\in[0,1]} \psi(w_\ell + \alpha(z_\ell - w_\ell))$ using exact minimization.

Since $z_\ell,\ w_\ell \in C$ and C is convex, we conclude from (4) that $w_{\ell+1} \in C$, thus the *CondG$_C$* method generates a sequence in C. Finally, (3) implies that $\langle v - w_\ell,\ z - w_\ell\rangle \le -s^*_\ell$, for all $z \in C$. Hence, considering the stopping criteria $-s^*_\ell \le \varphi_{\gamma,\theta,\lambda}(u, v, w_\ell)$, we conclude that *the output of CondG$_C$ method is a feasible inexact projection $w^+ = CondG_C\,(\varphi_{\gamma,\theta,\lambda}, u, v)$ of the point $v \in \mathbb{R}^n$ with respect $u \in C$ onto C*, i.e.,

$$\langle v - w^+,\ z - w^+\rangle \le \varphi_{\gamma,\theta,\lambda}(u, v, w^+), \qquad \forall\ z \in C. \tag{5}$$

Remark 1. *Let $C \subset \mathbb{R}^n$ be a closed and convex set. Thus, for $\varphi_{\gamma,\theta,\lambda}(u, v, w) \equiv 0$, it follows from Lemma 1 (i) together with* (5) *that $P_C(v) = CondG_C\,(\varphi_{\gamma,\theta,\lambda}, u, v)$, for all $v \in \mathbb{R}^n$ and $u \in C$. It is worth to noting that for projecting exact and inexact onto a compact convex set we need to assume that the LO oracle can be used effectively. For an example where LO oracle cannot be used effectively, see [35].*

The following three theorems state well-known convergence rate for classic conditional gradient method applied to problem $\min_{z\in C} \psi(z)$, for more details see [25, 30, 32]. Let us first remind some basic properties of the function ψ over the set C:

(i) $\psi(z) = \psi(w) + \langle \psi'(w),\ z - w\rangle + \|z - w\|^2/2$, for all $z, w \in C$;

(ii) $\psi(z) - \psi(w_*) \ge \|z - w_*\|^2/2$, for all $z \in C$;

(iii) $\|\psi'(z)\|^2 \ge 2\psi(w_*) = d^2(v, C)$, for all $z \in C$.

Since C is a compact set, we define the *diameter* of C by $d_C := \max_{z,w\in C} \|z-w\|$. The statement of the first convergence result is as follows, see the proof in [25].

Theorem 4. *Let $\{w_\ell\}$ be the sequence generated by Algorithm 1. Then, $\psi(w_\ell) - \psi(w_*) \le (8d_C^2)/\ell$, for all $\ell \ge 1$. Consequently, (by using item (ii) above) we have $\|w_\ell - w_*\| \le 4d_C/\sqrt{\ell}$, for all $\ell \ge 1$.*

Next we state of the second convergence result, its proof can be found in [30, Theorem 2].

Theorem 5. *Let $\{w_\ell\}$ be the sequence generated by Algorithm 1. Then, for each $\ell \ge 2$, there exists $1 \le \hat\ell \le \ell$ such that $-s^*_{\hat\ell} \le 2bd_C^2(1 + \delta)/(\ell + 2)$, where $b := 27/8$ and $\delta \ge 0$ is the accuracy to which the linear subproblem* (3) *are solved.*

We say that a convex set C is α_C*-strongly convex if, for any $z, w \in C$, $t \in [0, 1]$ and any vector $u \in \mathbb{R}^n$, it holds that $tz+(1-t)w+(\alpha_C/2)t(1-t)\|z-w\|^2u \in C$. The next result improves* the rate of convergence in Theorem 5 for α_C-strongly convex sets, its proof can be found in [25].

Theorem 6. *Assume that C is a α_C-strongly convex set. Let $\{w_\ell\}$ be generated by Algorithm 1 and set $q := \max\{1/2, 1 - \alpha_C d(v, C)/8\} < 1$. Then, $\psi(w_{\ell+1}) - \psi(w_*) \le q\,(\psi(w_\ell) - \psi(w_*))$, for all $\ell \ge 1$. Consequently, we have an exponentially convergence rate as follows $\psi(w_\ell) - \psi(w_*) \le (\psi(w_0) - \psi(w_*))q^\ell$, for all $\ell \ge 1$. Furthermore, we have $\|w_\ell - w_*\|^2 \le \sqrt{2(\psi(w_0) - \psi(w_*))}q^{\ell/2}$, for all $\ell \ge 1$.*

Let us present two useful properties of *CondG$_C$* method that will play important roles in the remainder of this paper.

Lemma 7. *Let $C \subset \mathbb{R}^n$ be convex, compact, and nonempty set. Let $v \in \mathbb{R}^n$, $u \in C$, $\gamma, \theta, \lambda \geq 0$ and $w^+ = CondG_C(\varphi_{\gamma,\theta,\lambda}, u, v)$. Then,*

$$\|w^+ - z\|^2 \leq \|v - z\|^2 + \frac{2\gamma + 2\lambda}{1 - 2\lambda}\|v - u\|^2 - \frac{1 - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2, \qquad z \in C, \tag{6}$$

for $0 \leq \lambda < 1/2$. Consequently, if $z = P_C(v)$ then

$$\|w^+ - P_C(v)\|^2 \leq \frac{2\gamma + 2\lambda}{1 - 2\lambda}\|v - u\|^2 + \frac{2\theta}{1 - 2\lambda}\|w^+ - v\|^2. \tag{7}$$

Proof. First, note that $\|w^+ - z\|^2 = \|v - z\|^2 - \|v - w^+\|^2 + 2\langle v - w^+, z - w^+\rangle$, for all $z \in C$. Since $w^+ = \mathrm{CondG}_C(\varphi_{\gamma,\theta,\lambda}, u, v)$, combining the last inequality with (5) and (2), after some algebraic manipulation, we obtain

$$\|w^+ - z\|^2 \leq \|v - z\|^2 + 2\gamma\|v - u\|^2 - (1 - 2\theta)\|v - w^+\|^2 + 2\lambda\|w^+ - u\|^2, \qquad z \in C. \tag{8}$$

On the other hand, $\|w^+ - u\|^2 = \|w^+ - v\|^2 + \|u - v\|^2 - 2\langle w^+ - v, u - v\rangle$, which implies that $\|w^+ - u\|^2 = \|u - v\|^2 - \|w^+ - v\|^2 + 2\langle v - w^+, u - w^+\rangle$. Since $w^+ = \mathrm{CondG}_C(\varphi_{\gamma,\theta,\lambda}, u, v)$ and $u \in C$, using (5) with $z = u$ and (2) with $w = w^+$, after some calculations, the last equation implies

$$\|w^+ - u\|^2 \leq \frac{1 + 2\gamma}{1 - 2\lambda}\|v - u\|^2 - \frac{1 - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2.$$

Combining last inequality with (8) we obtain (6). We proceed to prove (7). First note that letting $z = P_C(v)$ into inequality (6), the resulting inequality can be equivalently rewriting as follows

$$\|w^+ - P_C(v)\|^2 \leq \|v - P_C(v)\|^2 - \|w^+ - v\|^2 + \frac{2\gamma + 2\lambda}{1 - 2\lambda}\|v - u\|^2 - \frac{2\lambda - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2.$$

Thus, considering that $\|v - P_C(v)\|^2 - \|w^+ - v\|^2 \leq 0$ and $0 \leq \lambda < 1/2$ the desired inequality follows, which concludes the proof. □

Corollary 8. *Let $C, D \subset \mathbb{R}^n$ be convex, closed, and nonempty sets. Assume that the set C is compact and set $E := \{z \in C : \|z - P_D(z)\| = dist(C, D)\}$. Let $v \in D$, $u \in C$, $\gamma, \theta, \lambda \geq 0$, $w^+ = CondG_C(\varphi_{\gamma,\theta,\lambda}, u, v)$. Then, for each $\bar{z} \in E$ and $0 \leq \lambda < 1/2$, there holds*

$$\|w^+ - \bar{z}\|^2 \leq \|u - \bar{z}\|^2 + 2\langle u - v, P_D(\bar{z}) - v\rangle + \frac{2\gamma + 2\theta}{1 - 2\lambda}\|v - u\|^2 - \frac{2\lambda - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2.$$

Proof. First, applying (6) of Lemma 7 with $z = \bar{z}$ we obtain

$$\|w^+ - \bar{z}\|^2 \leq \|v - \bar{z}\|^2 + \frac{2\gamma + 2\lambda}{1 - 2\lambda}\|v - u\|^2 - \frac{1 - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2,$$

which is equivalently to

$$\|w^+ - \bar{z}\|^2 \leq \|v - \bar{z}\|^2 - \|w^+ - v\|^2 + \frac{2\gamma + 2\theta}{1 - 2\lambda}\|v - u\|^2 - \frac{2\lambda - 2\theta}{1 - 2\lambda}\|w^+ - v\|^2. \tag{9}$$

Second, note that $\|u - v\|^2 + \|v - \bar{z}\|^2 = \|u - \bar{z}\|^2 + 2\langle u - v, \bar{z} - v\rangle$. It turns out that $\langle u - v, \bar{z} - v\rangle = \langle u - v, \bar{z} - P_D(\bar{z})\rangle + \langle u - v, P_D(\bar{z}) - v\rangle$. Hence, from these two equalities, we obtain

$$\|u - v\|^2 + \|v - \bar{z}\|^2 = \|u - \bar{z}\|^2 + 2\langle u - v, \bar{z} - P_D(\bar{z})\rangle + 2\langle u - v, P_D(\bar{z}) - v\rangle. \tag{10}$$

Due to $\bar{z} \in E$, we have $\|\bar{z}-P_D(\bar{z})\| = dist(C, D) \leq \|w^+-v\|$ and, by Cauchy-Schwartz inequality, $2\langle u-v, \bar{z}-P_D(\bar{z})\rangle \leq 2\|u-v\|\|\bar{z}-P_D(\bar{z})\| \leq \|u-v\|^2+\|\bar{z}-P_D(\bar{z})\|^2$. These inequalities imply $2\langle u-v, \bar{z}-P_D(\bar{z})\rangle \leq \|u-v\|^2+\|w^+-v\|^2$, which combined with (10) yields

$$\|u-v\|^2+\|v-\bar{z}\|^2 \leq \|u-\bar{z}\|^2+\|u-v\|^2+\|w^+-v\|^2+2\langle u-v, P_D(\bar{z})-v\rangle.$$

This inequality is equivalente to $\|v-\bar{z}\|^2-\|w^+-v\|^2 \leq \|u-\bar{z}\|^2+2\langle u-v, P_D(\bar{z})-v\rangle$, which together with (9) yields the desired result. □

Let us end this section by presenting some examples of functions satisfying (2).

Example 1. *The functions* $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ *defined by* $\varphi_1(u, v, w) := \gamma\|v-u\|^2$, $\varphi_2(u, v, w) := \theta\|w-v\|^2$, $\varphi_3(u, v, w) := \gamma\theta\|v-u\|\|w-v\|$ *and* $\varphi_4(u, v, w) = \gamma\|v-u\|^2+\theta\|w-v\|^2+\lambda\|w-u\|^2$ *satisfy* (2).

4 The ACondG method with inexact projection onto one set

Next we present our first version of inexact alternating projection method to solve Problem (1), by using the CondG method to compute feasible inexact projections with respect to one of the sets in consideration, which will be named as *alternating conditional gradient-1 (ACondG-1)* method. For it, we assume that finding an exact projection onto the convex set B, not necessarily compact, is an easy task. We also assume that A is a convex and compact set and the projection onto it can be approximated by using the CondG$_A$ method. In this case, the *ACondG-1 method with feasible inexact projections onto one of the sets*, for solving the classic feasibility Problem (1), is formally defined as follows:

Algorithm 2: CondG method with inexact projection onto one set (ACondG-1)

Step 0. Let $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ be sequences of nonnegative real numbers and the associated function $\varphi_k := \varphi_{\gamma_k,\theta_k,\lambda_k}$, as defined in (2). Let $x_0 \in A$. If $x^0 \in B$, then **stop**. Otherwise, initialize $k \leftarrow 0$.

Step 1. Compute $P_B(x^k)$ and set the next iterate y^{k+1} as follows

$$y^{k+1} := P_B(x^k). \tag{11}$$

If $y_{k+1} \in A$, then **stop**.

Step 2. Use Agorithm 1 to compute CondG$_A$ $(\varphi_k, x^k, y^{k+1})$ and set the iterate x^{k+1} as follows

$$x^{k+1} := \mathrm{CondG}_A\,(\varphi_k, x^k, y^{k+1}). \tag{12}$$

If $x_{k+1} \in B$, then **stop**.

Step 3. Set $k \leftarrow k+1$, and go to **Step 1**.

First of all note that $x^k \in A$ and $y^k \in B$. Thus, if Algorithm 2 stops, it means that a point belonging to $A\cap B$ has been found. Therefore, we assume that $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ generated by Algorithm 2 are infinity sequences. In the following we will analyze Algorithm 2 first assuming that $A \cap B$ is nonempty and, then, considering that $A \cap B$ is empty.

4.1 The ACondG-1 method for two sets with nonempty intersection

In this section we assume that $A \cap B \neq \varnothing$. To proceed with the analysis of ACondG-1 method we need to *assume that the forcing sequences* $(\lambda_k)_{k\in\mathbb{N}}$*,* $(\gamma_k)_{k\in\mathbb{N}}$ *and* $(\theta_k)_{k\in\mathbb{N}}$ *satisfy*

$$\theta_k \leq \theta < 1/2, \qquad 2\gamma_k + 4\lambda_k \leq \sigma < 1 \qquad k = 0, 1, \ldots, \tag{13}$$

where θ and σ are positive real constants.

Theorem 9. *The sequences* $(x^k)_{k\in\mathbb{N}}$ *and* $(y^k)_{k\in\mathbb{N}}$ *converge to a point belonging to* $A \cap B \neq \varnothing$*.*

Proof. Take any $\bar{x} \in A \cap B$. From (11) we have $y^{k+1} = P_B(x^k)$. Then, applying Lemma 1 (ii) with $C = B$, $v = x^k$, and $z = \bar{x}$, we conclude

$$\|y^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \|y^{k+1} - x^k\|^2. \tag{14}$$

Using (12) and applying (6) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, $w^+ = x^{k+1}$, $z = \bar{x}$, and $C = A$, we obtain

$$\|x^{k+1} - \bar{x}\|^2 \leq \|y^{k+1} - \bar{x}\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|x^k - y^{k+1}\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2. \tag{15}$$

Therefore, the combination of (14) with (15) yields

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \frac{1 - 2\gamma_k - 4\lambda_k}{1 - 2\lambda_k}\|x^k - y^{k+1}\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2. \tag{16}$$

Hence, (13) and (16) imply that $(x^k)_{k\in\mathbb{N}}$ is Féjer convergent to $A \cap B \neq \varnothing$. Thus, by Lemma 3 (i), $(x^k)_{k\in\mathbb{N}}$ is bounded. Also, using Lemma 3 (ii), we conclude that $(\|x^k - \bar{x}\|)_{k\in\mathbb{N}}$ converges. Consequently, from (14) we obtain that $(y^k)_{k\in\mathbb{N}}$ is also bounded. Since A and B are closed sets, $(x^k)_{k\in\mathbb{N}} \subset A$, and $(y^k)_{k\in\mathbb{N}} \subset B$, all cluster cluster points of these sequences belong to the sets A and B, respectively. Now, using (16) together with (13), we obtain

$$0 < (1 - \sigma)\|x^k - y^{k+1}\|^2 \leq \|x^k - \bar{x}\|^2 - \|x^{k+1} - \bar{x}\|^2.$$

Since the right hand side of the last inequality converges to zero, $(\|y^{k+1} - x^k\|)_{k\in\mathbb{N}}$ also converges to zero. Thus all cluster points of $(x^k)_{k\in\mathbb{N}}$ are also clusters points of $(y^k)_{k\in\mathbb{N}}$, proving that there exists $\hat{x} \in A \cap B$ a cluster point of $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$. Therefore, from Lemma 3 (iii) we conclude that $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ converge to a same point in $A \cap B$. □

4.2 The ACondG-1 method for two sets with empty intersection

In the following we assume that the sets A and B have empty intersection, i.e., $A \cap B = \varnothing$. Since A is a compact set and the projection mapping P_B is continuous we have

$$\zeta := \sup\{\|x - P_B(u)\| : \ x, u \in A\} < +\infty. \tag{17}$$

Let us also *assume in this section that* $(\lambda_k)_{k\in\mathbb{N}}$*,* $(\gamma_k)_{k\in\mathbb{N}}$ *and* $(\theta_k)_{k\in\mathbb{N}}$ *are summable,*

$$\sum_{k\in\mathbb{N}} \gamma_k < \infty, \qquad \sum_{k\in\mathbb{N}} \theta_k < \infty, \qquad \sum_{k\in\mathbb{N}} \lambda_k < \infty. \tag{18}$$

In addition, we consider $0 \leq \lambda_k < 1/2$*, for all* $k = 0, 1, \ldots$*.*

Theorem 10. *The sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ converge, respectively, to $x^* \in A$ and $y^* \in B$ satisfying $\|x^* - y^*\| = dist(A, B)$.*

Proof. Since $(x^k)_{k\in\mathbb{N}} \subset A$ and $y^{k+1} = P_B(x^k)$, by using (17) we have $\|y^{k+1} - x^k\| \le \zeta$ and $\|x^{k+1} - y^{k+1}\| \le \zeta$. Thus, applying inequality (7) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_k = \varphi_{\gamma_k,\theta_k,\lambda_k}$, $w^+ = x^{k+1}$, and $C = A$, we obtain

$$\begin{aligned}\|x^{k+1} - P_A(y^{k+1})\|^2 &\le \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 + \frac{2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2 \\ &\le \frac{2\gamma_k + 2\theta_k + 2\lambda_k}{1 - 2\lambda_k}\zeta^2.\end{aligned}$$

Since (18) implies that $\lim_{k\to+\infty} \gamma_k = 0$, $\lim_{k\to+\infty} \theta_k = 0$, and $\lim_{k\to+\infty} \lambda_k = 0$, the last inequality yields

$$\lim_{k\to+\infty} \|x^{k+1} - P_A(y^{k+1})\| = 0. \tag{19}$$

On the other hand, applying Lemma 1 (ii) with $C = B$, $v = x^k$, and $z = y^k \in B$, and taking into account (11), we have

$$\|y^{k+1} - y^k\|^2 \le \|x^k - y^k\|^2 - \|y^{k+1} - x^k\|^2. \tag{20}$$

Applying inequality (6) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_k = \varphi_{\gamma_k,\theta_k,\lambda_k}$, $w^+ = x^{k+1}$, $z = x^k$ and $C = A$, and summing the obtained inequality with (20), after some algebraic manipulations, we obtain

$$\begin{aligned}\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 \le \|x^k - y^k\|^2 - \|x^{k+1} - y^{k+1}\|^2 \\ + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 + \frac{2\theta_k - 2\gamma_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2.\end{aligned}$$

Since $\|y^{k+1} - x^k\| \le \zeta$ and $\|x^{k+1} - y^{k+1}\| \le \zeta$, we conclude from the above inequality that

$$\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 \le \|x^k - y^k\|^2 - \|x^{k+1} - y^{k+1}\|^2 + \frac{2\gamma_k + 2\theta_k + 2\lambda_k}{1 - 2\lambda_k}\zeta^2.$$

Thus, using (18) and that $\lim_{k\to+\infty} \lambda_k = 0$, the last inequality yields

$$\sum_{k\in\mathbb{N}} \left(\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2\right) \le \|x^0 - y^0\|^2 + \zeta^2 \sum_{k\in\mathbb{N}} \frac{2\gamma_k + 2\theta_k + 2\lambda_k}{1 - 2\lambda_k} < +\infty,$$

which implies that both sequences $(\|x^{k+1} - x^k\|^2)_{k\in\mathbb{N}}$ and $(\|y^{k+1} - y^k\|^2)_{k\in\mathbb{N}}$ converge to zero. Hence, considering (11) and (19), we can apply Lemma 2 with $C = A$ and $D = B$, $v^k = x^k$ and $w^k = y^k$, for all $k = 0, 1, \ldots$, to conclude that each cluster point $\bar{x}$ of $(x^k)_{k\in\mathbb{N}}$ is a fixed point of P_AP_B, i.e., $\bar{x} = P_AP_B(\bar{x})$, $\lim_{k\to\infty} \|x^k - P_B(x^k)\| = dist(A, B)$ and $\lim_{k\to\infty}(x^k - P_B(x^k)) = P_{A-B}(0)$.

Now, we are going to prove that the whole sequence $(x^k)_{k\in\mathbb{N}}$ converges. For that, consider the set $E = \{x \in A : \|x - P_B(x)\| = dist(A, B)\}$. We already proved that all clusters point of $(x^k)_{k\in\mathbb{N}}$ belong to E. Applying Corollary 8 with $v = y^{k+1}$, $u = x^k$, $w^+ = x^{k+1}$, $\bar{z} = \bar{x} \in E$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_k = \varphi_{\gamma_k,\theta_k,\lambda_k}$, $C = A$, and $D = B$, we obtain

$$\begin{aligned}\|x^{k+1} - \bar{x}\|^2 \le \|x^k - \bar{x}\|^2 + 2\langle x^k - y^{k+1}, P_B(\bar{x}) - y^{k+1}\rangle \\ + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{2\lambda_k - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2.\end{aligned}$$

Since $y^{k+1} = P_B(x^k)$, applying Lemma 1 (i) with $v = x^k$ and $z = P_B(\bar{x})$ we have

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{2\lambda_k - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2.$$

Therefore, due to $\|y^{k+1} - x^k\| \leq \zeta$, $\|x^{k+1} - y^{k+1}\| \leq \zeta$ and $0 \leq \lambda_k < 1/2$, it follows that

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + \zeta^2 \frac{2\gamma_k + 2\theta_k + 2\lambda_k}{1 - 2\lambda_k}. \tag{21}$$

By using (18) and taking into account that $\lim_{k\to+\infty} \lambda_k = 0$, we obtain

$$\sum_{k\in\mathbb{N}} \zeta^2 \frac{2\gamma_k + 2\theta_k + 2\lambda_k}{1 - 2\lambda_k} < \infty,$$

which combined with (21) implies that $(x^k)_{k\in\mathbb{N}}$ is quasi-Féjer convergent to the set E. Since the sequence $(x^k)_{k\in\mathbb{N}}$ has a cluster point belonging to E, it follows from Lemma 3 that the whole sequence converge to a point $x^* \in E$. Hence, it follows from (11) that $\lim_{k\to+\infty} y^{k+1} = P_B(x^*)$. Therefore, setting $y^* = P_B(x^*)$ and due to $x^* = P_A P_B(x^*)$ we also have $y^* = P_B P_A(y^*)$. By using [12, Theorem 2] we obtain that $\|y^* - P_A(y^*))\| = dist(A, B)$, which concludes the proof. $\square$

5 The ACondG method with inexact projections onto two sets

In this section, we present our second version of inexact alternating projection method to solve Problem (1), by using the CondG method for computing feasible inexact projections with respect to both sets in consideration. This method will be named as *alternating conditional gradient-2 (ACondG-2)* method. Let us assume that A and B are convex and compact sets. The ACondG-2 method, is formally defined as follows:

Algorithm 3: ACondG method with inexact projection onto two sets (ACondG-2)

Step 0. Let $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ be sequences of nonnegative real numbers and the associated function $\varphi_k := \varphi_{\gamma_k,\theta_k,\lambda_k}$, as defined in (2). Let $x_0 \in A$, $y_0 \in B$. If $x^0 \in B$ or $y^0 \in A$, then **stop**. Otherwise, initialize $k \leftarrow 0$.

Step 1. Using Agorithm 1, compute $\text{CondG}_B\,(\varphi_k, y^k, x^k)$ and set the next iterate y^{k+1} as

$$y^{k+1} := \text{CondG}_B\,(\varphi_k, y^k, x^k). \tag{22}$$

If $y_{k+1} \in A$, then **stop**.

Step 2. Using Agorithm 1, compute $\text{CondG}_A\,(\varphi_k, x^k, y^{k+1})$ and set the next iterate x^{k+1} as

$$x^{k+1} := \text{CondG}_A\,(\varphi_k, x^k, y^{k+1}).$$

If $x^{k+1} \in B$, then **stop**.

Step 3. Set $k \leftarrow k + 1$, and go to **Step 1**.

As for Algorithm 2, if Algorithm 3 stops, it means that a point belonging to $A \cap B$ has been found. Therefore, hereafter we assume that $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ generated by Algorithm 3 are infinity sequences. We will proceed with the convergence analysis of ACondG-2 method by considering the cases where $A \cap B \neq \varnothing$ and $A \cap B = \varnothing$.

5.1 The ACondG-2 method for two sets with nonempty intersection

Let us assume that A and B have nonempty intersection, that is, $A \cap B \neq \varnothing$. Additionally, suppose that the forcing sequences $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ satisfy

$$\theta_k \le \bar{\theta} < 1/4, \qquad 2\gamma_k + 3\lambda_k \le \sigma < 1/2, \qquad 2\gamma_k + 2\theta_k + 2\lambda_k \le \rho < 1, \tag{23}$$

for all $k = 0, 1, \ldots$, where $\bar{\theta}$, σ and ρ are positive real constants.

Theorem 11. *The sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ converge to a point belonging to set $A \cap B \neq \varnothing$.*

Proof. Let $\bar{x} \in A \cap B$ and $k \in \mathbb{N}$. Applying (6) of Lemma 7 with $v = x^k$, $u = y^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, λ_k, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, $w^+ = y^{k+1}$, $z = \bar{x}$, and $C = B$, we have

$$\|y^{k+1} - \bar{x}\|^2 \le \|x^k - \bar{x}\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|x^k - y^k\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2. \tag{24}$$

On the other hand, applying (6) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, $w^+ = x^{k+1}$, $z = \bar{x}$, and $C = A$, we obtain

$$\|x^{k+1} - \bar{x}\|^2 \le \|y^{k+1} - \bar{x}\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2. \tag{25}$$

Combining inequalities (24) and (25) we conclude that

$$\begin{aligned}\|x^{k+1} - \bar{x}\|^2 \le \|x^k - \bar{x}\|^2 &- \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|y^{k+1} - x^{k+1}\|^2 \\ &+ \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^k - x^k\|^2 - \frac{1 - 2\gamma_k - 2\theta_k - 2\lambda_k}{1 - 2\lambda_k}\|x^k - y^{k+1}\|^2. \end{aligned} \tag{26}$$

Due to second condition in (23) we have $(2\gamma_k + 2\lambda_k)/(1 - 2\lambda_k) < 1/2$. Taking into account (23) and (26) we obtain

$$\|x^{k+1} - \bar{x}\|^2 + \frac{1}{2}\|x^{k+1} - y^{k+1}\|^2 \le \|x^k - \bar{x}\|^2 + \frac{1}{2}\|x^k - y^k\|^2. \tag{27}$$

In particular, (27) implies that the sequence $(\|x^k - \bar{x}\|^2 + \frac{1}{2}\|x^k - y^k\|^2)_{k\in\mathbb{N}}$ is non-increasing. Hence, it converges and, moreover, $(x^k)_{k\in\mathbb{N}}$ is bounded. We also have from (23) and (26) that

$$\|x^k - y^{k+1}\|^2 \le \frac{1}{\rho}\Big[\Big(\|x^k - \bar{x}\|^2 + \frac{1}{2}\|x^k - y^k\|^2\Big) - \Big(\|x^{k+1} - \bar{x}\|^2 + \frac{1}{2}\|x^{k+1} - y^{k+1}\|^2\Big)\Big]. \tag{28}$$

In its turn, applying (6) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, $w^+ = x^{k+1}$, $z = x^k$ and $C = A$, we have

$$\|x^{k+1} - x^k\|^2 \le \|y^{k+1} - x^k\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2.$$

Thus, by using the two first inequalities in (23), we have $\|x^{k+1}-x^k\| \leq 2\|y^{k+1}-x^k\|$, which after applying the triangle inequality, yields $\|x^{k+1}-y^{k+1}\| \leq 3\|x^k-y^{k+1}\|$. Hence, since (28) implies $\sum_{k\in\mathbb{N}} \|x^k - y^{k+1}\|^2 < +\infty$, we obtain $\sum_{k\in\mathbb{N}} \|x^k - y^k\|^2 < +\infty$. In particular, this inequality implies that $(\|x^k - y^k\|)_{k\in\mathbb{N}}$ converges to zero. Hence, $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ has the same cluster points. Taking into account that $(x^k)_{k\in\mathbb{N}} \subset A$ and $(y^k)_{k\in\mathbb{N}} \subset B$, we conclude that all cluster points of $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ are in $A \cap B \neq \varnothing$. Finally, the combination of (26) with (23) implies

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + \frac{3}{2}\|x^k - y^k\|^2,$$

and considering that $\sum_{k\in\mathbb{N}} \|x^k - y^k\|^2 < +\infty$, we conclude that $(x^k)_{k\in\mathbb{N}}$ is quasi-Féjer convergence to $A \cap B \neq \varnothing$. Since all clusters point of $(x^k)_{k\in\mathbb{N}}$ belongs to $A \cap B$, Lemma 3*(iii)* implies that it converges to a point in $A\cap B$. Therefore, due to the cluster points of $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ are the same, the results follows and the proof is concluded. □

5.2 The ACondG-2 method for two sets with empty intersection

Now, we assume that the sets A and B have empty intersection, that is, $A \cap B = \varnothing$. Since A and B are bounded we have

$$\omega := \sup\{\|x - y\| : \ x \in A, \ y \in B\} < +\infty.$$

As in section 4.2, we also assume that $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ are summable, i.e., satisfy (18), and that $0 \leq \lambda_k < 1/2$, for all $k = 0, 1, \ldots$.

Theorem 12. *The sequences $(x^k)_{k\in\mathbb{N}}$ and $(y^k)_{k\in\mathbb{N}}$ converge respectively to $x^* \in A$ and $y^* \in B$ which satisfy $\|x^* - y^*\| = dist(A, B)$.*

Proof. Let $x \in A$ and $y \in B$. Applying (7) of Lemma 7 with $v = x^k$, $u = y^k$, $w^+ = y^{k+1}$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, and $C = B$, we have

$$\|y^{k+1} - P_B(x^k)\|^2 \leq \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|x^k - y^k\|^2 + \frac{2\theta_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2.$$

Considering that $(x^k)_{k\in\mathbb{N}} \subset A$ and $(y^k)_{k\in\mathbb{N}} \subset B$, we have $\|x^k - y^k\| \leq \omega$ and $\|y^{k+1} - x^k\| \leq \omega$. Moreover, (18) implies that $\lim_{k\to+\infty} \gamma_k = 0$, $\lim_{k\to+\infty} \theta_k = 0$, and $\lim_{k\to+\infty} \lambda_k = 0$. Thus, from the above inequality, we obtain

$$\lim_{k\to+\infty} \|y^{k+1} - P_B(x^k)\|^2 = 0. \tag{29}$$

Applying again (7) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $w^+ = x^{k+1}$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$ and $C = A$, we can also conclude that

$$\lim_{k\to+\infty} \|x^{k+1} - P_A(y^{k+1})\|^2 = 0. \tag{30}$$

On the other hand, applying (6) of Lemma 7 with $v = x^k$, $u = y^k$, $w^+ = y^{k+1}$ $z = y^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, and $C = B$, it follows that

$$\|y^{k+1} - y^k\|^2 \leq \|x^k - y^k\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|x^k - y^k\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2.$$

Now, applying (6) of Lemma 7 with $v = y^{k+1}$, $u = x^k$, $w^+ = x^{k+1}$, $z = x^k$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, and $C = A$, we obtain

$$\|x^{k+1} - x^k\|^2 \leq \|y^{k+1} - x^k\|^2 + \frac{2\gamma_k + 2\lambda_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2.$$

Summing the above two previous inequalities we conclude

$$\begin{aligned}\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 \leq \frac{1 + 2\gamma_k}{1 - 2\lambda_k}\|x^k - y^k\|^2 \\ - \frac{1 - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2 + \frac{2\theta_k + 2\gamma_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2.\end{aligned}$$

Thus, considering that $\|x^{k+1} - y^{k+1}\| \leq \omega$, $\|x^k - y^k\| \leq \omega$, and $\|y^{k+1} - x^k\| \leq \omega$ and after some algebraic manipulations, we have

$$\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 \leq \|x^k - y^k\|^2 - \|x^{k+1} - y^{k+1}\|^2 + \frac{4\gamma_k + 4\theta_k + 2\lambda_k}{1 - 2\lambda_k}\omega^2.$$

Consequently, using (18) and that $\lim_{k\to+\infty} \lambda_k = 0$, we obtain

$$\sum_{k\in\mathbb{N}} \left(\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2\right) \ \leq \ \|x^0 - y^0\|^2 + \omega^2 \sum_{k\in\mathbb{N}} \frac{4\gamma_k + 4\theta + 2\lambda_k}{1 - 2\lambda_k} \ < \ +\infty, \quad (31)$$

which implies that $(\|x^{k+1} - x^k\|^2)_{k\in\mathbb{N}}$ and $(\|y^{k+1} - y^k\|^2)_{k\in\mathbb{N}}$ converge to zero. Hence, considering (29) and (30), we can apply Lemma 2 with $C = A$ and $D = B$, $v^k = x^k$ and $w^k = y^k$, for all $k = 0, 1, \ldots$, to conclude that each cluster point $\bar{x}$ of $(x^k)_{k\in\mathbb{N}}$ is a fixed point of $P_A P_B$, i.e., $\bar{x} = P_A P_B(\bar{x})$, $\lim_{k\to\infty} \|x^k - P_B(x^k)\| = dist(A, B)$ and $\lim_{k\to\infty}(x^k - P_B(x^k)) = P_{A-B}(0)$.

Now, we are going to prove that the whole sequence $(x^k)_{k\in\mathbb{N}}$ converges. For that, consider the set $E = \{x \in A : \|x - P_B(x)\| = dist(A, B)\}$. We already proved that all clusters point of $(x^k)_{k\in\mathbb{N}}$ belong to E. Take $\bar{x} \in E$. Applying Corollary 8 with $v = y^{k+1}$, $u = x^k$, $w^+ = x^{k+1}$, $\bar{z} = \bar{x}$, $\gamma = \gamma_k$, $\theta = \theta_k$, $\lambda = \lambda_k$, $\varphi_{\gamma,\theta,\lambda} = \varphi_k$, $C = A$ and $D = B$, we obtain

$$\begin{aligned}\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + 2\langle x^k - y^{k+1}, P_B(\bar{x}) - y^{k+1}\rangle \\ + \frac{2\gamma_k + 2\theta_k}{1 - 2\lambda_k}\|y^{k+1} - x^k\|^2 - \frac{2\lambda_k - 2\theta_k}{1 - 2\lambda_k}\|x^{k+1} - y^{k+1}\|^2. \quad (32)\end{aligned}$$

Now, by using (22) we have $y^{k+1} = \mathrm{CondG}_B(\varphi_k, y^k, x^k)$. Hence, it follows from (5) that $\langle x^k - y^{k+1}, P_B(\bar{x}) - y^{k+1}\rangle \leq \varphi_{\gamma_k,\theta_k,\lambda_k}(y^k, x^k, y^{k+1})$. Then, from (2) we have

$$\langle x^k - y^{k+1}, P_B(\bar{x}) - y^{k+1}\rangle \leq \gamma_k\|x^k - y^k\|^2 + \theta_k\|y^{k+1} - x^k\|^2 + \lambda_k\|y^{k+1} - y^k\|^2.$$

Therefore, due to $\|x^k - y^k\| \leq \omega$, $\|y^{k+1} - x^k\| \leq \omega$, $\|x^{k+1} - y^{k+1}\| \leq \omega$ and $0 \leq \lambda_k < 1/2$, it follows from the last inequality and (32) that

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + \omega^2\left(2\gamma_k + 2\theta_k + \frac{2\gamma_k + 4\theta_k}{1 - 2\lambda_k}\right) + \|y^{k+1} - y^k\|^2. \quad (33)$$

By using (18), (31) and that $\lim_{k\to+\infty} \lambda_k = 0$, we obtain

$$\sum_{k\in\mathbb{N}} \left[\omega^2\left(2\gamma_k + 2\theta_k + \frac{2\gamma_k + 4\theta_k}{1 - 2\lambda_k}\right) + \|y^{k+1} - y^k\|^2\right] < \infty,$$

which combined with (33) implies that $(x^k)_{k\in\mathbb{N}}$ is quasi-Féjer convergent to E. Since the sequence $(x^k)_{k\in\mathbb{N}}$ has a cluster point belonging to E, it follows that the whole sequence converges to a point $x^* \in E$. Finally, we also know that $(x^k - y^k)_{k\in\mathbb{N}}$ converges. Considering that $y^k = x^k + (y^k - x^k) \in B$, for all $k = 0, 1, \ldots$, we conclude that $(y^k)_{k\in\mathbb{N}}$ also converges to a point $y^* \in B$. Hence, it follows from (29) that $\lim_{k\to+\infty} y^k = P_B(x^*)$. Therefore, $y^* = P_B(x^*)$ and due to $x^* = P_A P_B(x^*)$ we also have $y^* = P_B P_A(y^*)$ and, by using [12, Theorem 2], we obtain $\|y^* - P_A(y^*))\| = dist(A, B)$, which concludes the proof. $\square$

6 Numerical examples

The purpose of this section is illustrate the practical behavior and demonstrate the potential advantages of the ACondG-1 and ACondG-2 algorithms over their *exact* counterparts, i.e., Algorithms 2 and 3 where the projections are computed up to a small prescribed tolerance. For future reference, we will call the exact schemes by ExactAlg2 and ExactAlg3 corresponding to ACondG-1 and ACondG-2, respectively. All codes were implemented in Fortran 90 and are freely available at `http://lfprudente.ime.ufg.br/up/948/o/ACondG.tar.gz`.

In our implementations, sets A and B are described in the general form

$$A := \{z \in \mathbb{R}^n \colon h_A(z) = 0,\ g_A(z) \le 0\}, \qquad B := \{z \in \mathbb{R}^n \colon h_B(z) = 0,\ g_B(z) \le 0\},$$

where $h_A \colon \mathbb{R}^n \to \mathbb{R}^{m_A}$, $g_A \colon \mathbb{R}^n \to \mathbb{R}^{p_A}$, $h_B \colon \mathbb{R}^n \to \mathbb{R}^{m_B}$, and $g_B \colon \mathbb{R}^n \to \mathbb{R}^{p_B}$ are continuously differentiable functions. The feasibility violations at a given point $z \in \mathbb{R}^n$ with respect to sets A and B are measured, respectively, by

$$c_A(z) := \max\{\|h_A(z)\|_\infty, \|V_A(z)\|_\infty\} \quad \text{and} \quad c_B(z) := \max\{\|h_B(z)\|_\infty, \|V_B(z)\|_\infty\},$$

where

$$[V_A(z)]_i = \max\{0, [g_A(z)]_i\}, \qquad [V_B(z)]_j = \max\{0, [g_B(z)]_j\},$$

$i = 1, \ldots, p_A$, and $j = 1, \ldots, p_B$. The algorithms are successfully stopped at iteration k, declaring that a feasible point was found, if

$$c_B(x^{k+1}) \le \varepsilon_{feas} \quad \text{or} \quad c_A(y^{k+1}) \le \varepsilon_{feas},$$

where $\varepsilon_{feas} > 0$ is an algorithmic parameter. We also consider a stopping criterion related to lack of progress: the algorithms terminate if, for two consecutive iterations, it holds that

$$\|x^{k+1} - x^k\|_\infty \le \varepsilon_{lack} \quad \text{and} \quad \|y^{k+1} - y^k\|_\infty \le \varepsilon_{lack},$$

where $\varepsilon_{lack} > 0$ is also an algorithmic parameter. Note that the latter criterion should be satisfied if the intersection between sets A and B is empty.

In the CondG scheme given by Algorithm 1, for computing the optimal solution z_ℓ at Step 1, we used the software Algencan [7], an augmented Lagrangian code for general nonlinear optimization programming. Algencan was also used to compute the projections of the exact schemes. In the latter case, we solve the convex quadratic problem given by $\min_{z\in C} \frac{1}{2}\|z - v\|^2$, considering a tolerance of 10^{-8} for the optimality (measured by the Karush–Kuhn–Tucker system).

Function $\varphi_{\gamma,\theta,\lambda}$ related to the degree of inexactness of the projections was set to be equal to the right hand side of (2). The forcing sequences $(\gamma_k)_{k\in\mathbb{N}}$, $(\theta_k)_{k\in\mathbb{N}}$, and $(\lambda_k)_{k\in\mathbb{N}}$ are defined in an adaptive manner. We first choose γ_0, θ_0, and λ_0 satisfying either condition (13) or (23) for

ACondG-1 and ACondG-2 algorithms, respectively. For the subsequent iterations if, between two consecutive iterations, enough progress is observed in terms of feasibility with respect to the sets A or B, the parameters are not updated. Otherwise, the parameters are decreased by a fixed factor. This means that when a lack of progress is verified, the forcing parameters are decreased requiring more accurate projections. Formally, for $k \geq 1$, we set

$$(\gamma_k, \theta_k, \lambda_k) := \begin{cases} (\gamma_{k-1}, \theta_{k-1}, \lambda_{k-1}), \text{ if } c_B(x^k) \leq \tau c_B(x^{k-1}) \text{ or } c_A(y^k) \leq \tau c_A(y^{k-1}), \\ \delta(\gamma_{k-1}, \theta_{k-1}, \lambda_{k-1}), \text{ otherwise,} \end{cases}$$

where $\tau, \delta \in (0,1)$ are algorithmic parameters. Since $(\gamma_k)_{k\in\mathbb{N}}$, $(\theta_k)_{k\in\mathbb{N}}$, and $(\lambda_k)_{k\in\mathbb{N}}$ are non-increasing sequences, either condition (13) or (23), according to the chosen method, holds for all $k \geq 0$. Moreover, observe that if $A \cap B = \varnothing$, then there exists $k_0 \in \mathbb{N}$ such that $(\gamma_k, \theta_k, \lambda_k) = \delta(\gamma_{k-1}, \theta_{k-1}, \lambda_{k-1})$ for all $k > k_0$. Thus, $(\gamma_k, \theta_k, \lambda_k) = \delta^{k-k_0}(\gamma_{k_0}, \theta_{k_0}, \lambda_{k_0})$ for all $k > k_0$. As consequence, $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ are summable, because $\delta \in (0,1)$.

In our tests we set $\varepsilon_{feas} = \varepsilon_{lack} = 10^{-8}$, $\gamma_0 = 0.1 - \varepsilon_{feas}$, $\theta_0 = \lambda_0 = 0.2 - \varepsilon_{feas}$, $\tau = 0.9$, and $\delta = 0.1$.

6.1 ACondG-1 algorithm

In this subsection we consider the problem of finding a point in the intersection of a region delimited by an ellipse and a half-plane. Given $z_0^A \in \mathbb{R}^2$, $\theta \in [-\pi, \pi]$, $a, b > 0$, and defining

$$D(a,b) := \begin{bmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{bmatrix} \quad \text{and} \quad R(\theta) := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{34}$$

we take

$$A = \left\{ z \in \mathbb{R}^2\colon\ \left\langle D(a,b)R(\theta)(z - z_0^A),\ R(\theta)(z - z_0^A) \right\rangle - 1 \leq 0 \right\}. \tag{35}$$

Let $\beta \in \mathbb{R}$ a parameter and define

$$B = \left\{ z \in \mathbb{R}^2\colon\ -[z]_1 + \beta \leq 0 \right\}. \tag{36}$$

Clearly, there exists an explicit expression for the projection onto B. It is worth mentioning that ellipses satisfy the assumptions of Theorem 6; see [25, Lemma 2]. In our tests, we set $z_0^A = [0,0]^T$, $\theta = -\pi/4$, $a = 2$, and $b = 1/5$ in (35) for defining A, and considered different values of β in (36) for B. Parameter β allows to control the existence of points in the intersection of sets A and B.

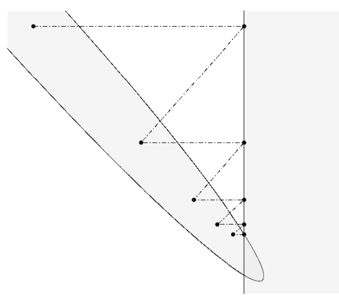
Table 1 shows the performance of ACondG-1 and ExactAlg2 algorithms on eight instances of the considered problem, while Figure 1 illustrates some "solutions". The initial point x^0 of each instance was taken to be the center z_0^A. In the table, the first column contains the considered values of β and "$A \cap B$" informs whether the intersection of sets A and B is empty or not. Observe that in the first four instances there are points at the intersection of A and B, while in the last four the intersection is empty. For each algorithm, "SC" informs the satisfied stopping criterion where "C" denotes *convergence* meaning that the algorithm found a point in $A \cap B$ and "L" means that the algorithms stopped due to lack of progress, "it" is the number of iterations, and "$\min\{c_B(x^*), c_A(y^*)\}$" is the smallest feasibility violation with respect to sets A and B at the final iterates.

In the first four instances, the CondG-1 algorithm converged to a solution. In fact, in these cases, we emphasize that a feasible point has been found, *not just* an infeasible point satisfying

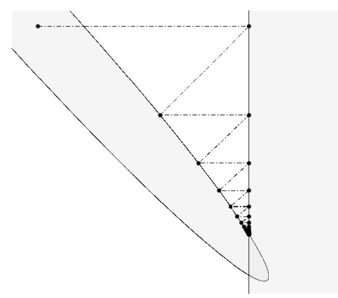
		ACondG-1			ExactAlg2		
β	$A \cap B$	SC	it	$\min\{c_B(x^*), c_A(y^*)\}$	SI	it	$\min\{c_B(x^*), c_A(y^*)\}$
1.30		C	5	0.00D+00	L	46	1.47D-08
1.35	$\neq \varnothing$	C	20	0.00D+00	L	53	1.44D-08
1.40		C	29	0.00D+00	L	78	2.11D-08
1.42		C	120	0.00D+00	L	348	5.67D-08
1.43		L	45	8.73D-03	L	110	8.73D-03
1.45	$= \varnothing$	L	24	2.87D-02	L	49	2.87D-02
1.50		L	19	7.87D-02	L	28	7.87D-02
1.60		L	9	1.79D-01	L	19	1.79D-01

Table 1: Performance of ACondG-1 and ExactAlg2 algorithms on eight instances of the problem of finding a point in $A \cap B$, where the sets are given as (35) and (36).

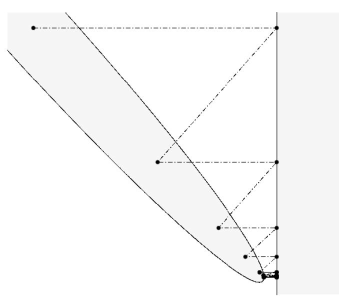
(a) ACondG-1 ($\beta = 1.30$)



(b) ExactAlg2 ($\beta = 1.30$)



(c) ACondG-1 ($\beta = 1.50$)



(d) ExactAlg2 ($\beta = 1.50$)

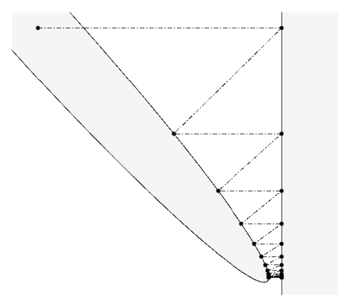


Figure 1: Behavior of ACondG-1 and ExactAlg2 algorithms on the problem of finding a point in $A \cap B$, where the sets are given as (35) and (36) with $\beta = 1.30$ and $\beta = 1.50$.

the prescribed feasibility tolerance. On the other hand, ExactAlg2 failed to achieve the required feasibility tolerance $\varepsilon_{feas} = 10^{-8}$ in all instances, stopping for lack of progress. For ExactAlg2, we point out that the feasibility violation measure $\min\{c_B(x^{k+1}), c_A(y^{k+1})\}$ arrived $\mathcal{O}(10^{-8})$ after 39, 45, 69, and 331 iterations for $\beta = 1.30$, 1.35, 1.40, and 1.42, respectively. This mean that ACondG-1 used 87.2%, 55.6%, 58.0%, and 63, 4% fewer iterations for finding a feasible point than ExactAlg2 for achieve $\mathcal{O}(10^{-8})$ in the feasibility violation measure. This seems surprising at first, because in most applications, an inexact algorithm has a low cost per iteration compared to its exact version, although the overall number of iterations tends to increase. Figures 1(a) and (b) show the behavior of ACondG-1 and ExactAlg2 algorithms for $\beta = 1.30$. As can be seen, the iterates of ExactAlg2 are always at the boundary of set A, getting stuck at infeasible points *near* the intersection of the sets. In contrast, ACondG-1 goes into the interior of set A approaching the intersection faster and reaching a feasible point.

In the four infeasible instances, as expected, both algorithms stopped for lack of progress reaching the same feasibility violation measure. However, as shown in Table 1, ACondG-1 required, on average, 48.7% fewer iterations than ExactAlg2 to stop. Figures 1(c) and (d) show the behavior of ACondG-1 and ExactAlg2 algorithms for $\beta = 1.50$. Observe that, as k grows, the forcing parameters go to zero and the iterations of ACondG-1 become exact ones. This can be seen by noting that the iterates x^k, for large k, belong to the boundary of the ellipse.

6.2 ACondG-2 algorithm

Now we consider the problem of finding a point in the intersection of regions delimited by two ellipses. Let A as in section 6.1. For defining B, let $z_0^B \in \mathbb{R}^2$ and set

$$B = \left\{ z \in \mathbb{R}^2 \colon \left\langle D(c,d)R(\vartheta)(z - z_0^B),\ R(\vartheta)(z - z_0^B) \right\rangle - 1 \leq 0 \right\}, \tag{37}$$

where $D(\cdot,\cdot)$ and $R(\cdot)$ are given as (34), $\vartheta = \pi/3$, $c = 2$, and $d = 2/5$.

In our tests, we defined $z_0^B = [\cdot, 1/2]^T$ and considered different values for the first coordinate $[z_0^B]_1$. As for parameter β in section 6.1, $[z_0^B]_1$ can be used to determine whether or not there are points at the intersection of A and B. Table 2 shows the performance of ACondG-2 and ExactAlg3 algorithms on eight instances of the problem corresponding to the values of $[z_0^B]_1$ given in the first column. The remaining columns are as defined for Table 1. In each instance, the initial points were taken to be the centers of the ellipses, i.e., $x^0 = z_0^A$ and $y^0 = z_0^B$. Figure 2 shows the behavior of the methods in the particular cases where $[z_0^B]_1 = 2.30$ and $[z_0^B]_1 = 2.50$.

		ACondG-2			ExactAlg3		
$[z_0^B]_1$	$A \cap B$	SC	it	$\min\{c_B(x^*), c_A(y^*)\}$	SI	it	$\min\{c_B(x^*), c_A(y^*)\}$
2.30	$\neq \varnothing$	C	2	0.00D+00	L	40	2.71D-08
2.35		C	2	0.00D+00	L	127	4.60D-08
2.357		C	8	0.00D+00	L	398	7.63D-08
2.358		C	155	0.00D+00	L	699	1.06D-07
2.359	$= \varnothing$	L	724	1.50D-04	L	8378	7.31D-05
2.36		L	304	1.01D-03	L	1091	1.00D-03
2.40		L	23	4.01D-02	L	57	4.01D-02
2.50		L	15	1.59D-01	L	25	1.59D-01

Table 2: Performance of ACondG-2 and ExactAlg3 algorithms on eight instances of the problem of finding a point in $A \cap B$, where the sets are given as (35) and (37).

In the four feasible instances, ACondG-2 found a point in the intersection of A and B while ExactAlg3 stopped due to lack of progress. As for the exact scheme in the previous section, ExactAlg3 got stuck at infeasible points *near* the intersection of the sets, see Figure 2(b). We report that, with respect to ExactAlg3, the feasibility violation measure $\min\{c_B(x^{k+1}), c_A(y^{k+1})\}$ arrived $\mathcal{O}(10^{-8})$ after 35, 118, and 386 iterations for the first three instances, respectively, and $\mathcal{O}(10^{-7})$ after 526 iterations for the fourth instance. In its turn, as can be seen from Table 2, ACondG-2 found a feasible point in 2, 2, 8, and 155 iterations, respectively, showing the *huge* performance difference between the methods for this class of problem. As suggested by Figure 2(a), since the iterates lie in the interior of the two sets, ACondG-2 may be able to find a feasible point very quickly.

(a) ACondG-2 ($[z_0^B]_1 = 2.30$) (b) ExactAlg3 ($[z_0^B]_1 = 2.30$)

(c) ACondG-2 ($[z_0^B]_1 = 2.50$) (d) ExactAlg3 ($[z_0^B]_1 = 2.50$)

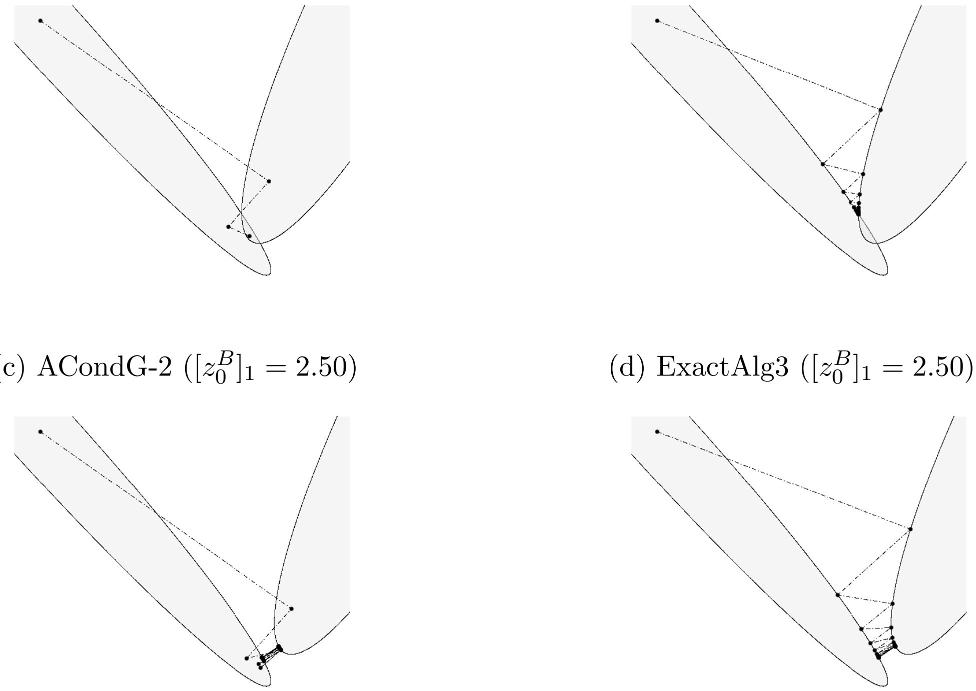


Figure 2: Behavior of ACondG-2 and ExactAlg3 algorithms on the problem of finding a point in $A \cap B$, where the sets are given as (35) and (37) with $[z_0^B]_1 = 2.30$ and $[z_0^B]_1 = 2.50$.

For the infeasible instances, the feasibility violation measure arrived, respectively, $\mathcal{O}(10^{-4})$, $\mathcal{O}(10^{-3})$, $\mathcal{O}(10^{-2})$, and $\mathcal{O}(10^{-1})$ after 33, 5, 3, and 2 iterations for ACondG-2 and after 82, 15, 5, and 2 for ExactAlg3. This shows that, for the chosen set of problems, ACondG-2 approaches the nearest region between A and B faster than ExactAlg3, see Figures 2(c) and (d). On average, ACondG-2 required 65.7% fewer iterations than ExactAlg3 for stopping due to lack of progress. On the other hand, in some instances, ExactAlg3 obtained a final iterate with a smaller feasibility violation measure. This can be explained by the fact that algorithms use different numerical approaches to calculate projections.

Last but not least, the performance of the methods presented throughout the numerical results section should be taken as an illustration of the capabilities of the introduced methods with respect to their exact counterparts, taking into account that they correspond to small problems with specific structures. More precise conclusions should be made after numerical experiments using problems of different classes and scales.

7 Conclusions

In the present paper we propose a new method that we call ACondG method to solve Problem (1) by combining CondG method with the alternate projections method. Let us state some questions related to the ACondG method that deserve to be investigated.

As suggested by the numerical experiments, the proposed method seems promising. Let us highlight some aspects observed during the numerical tests. In the chosen set of test problems, the inexact methods performed fewer iterations than the exact ones. In particular, whenever the intersection of the involved sets has a nonempty interior, the methods converged in a finite number of iterations. These phenomena deserve further investigations. Given that the "exact algorithms" are actually inexact, it would be interesting to compare the performance of the Algorithm 2 and Algorithm 3 with the other algorithms in terms of time, and to see if depending on how accurate you compute the "exact projections" you can become competitive with the Algorithm 2 and Algorithm 2.

An interesting theoretical aspect that also deserves to be investigated is the possibility of unifying Theorems 9 and 10 for Algorithm 2, and also Theorems 11 and 12 for Algorithm 3. In addition, to obtain less stringent forcing parameters requirements and new bounds to the tolerance function.

The $\varphi_{\gamma,\theta,\lambda}$ parameter controls the accuracy to which the projections are computed, see **Step 2** in Algorithm 1. Setting a value of $\varphi_{\gamma,\theta,\lambda}$ that is very small, it heavily impacts the number of inner iterations needed to satisfy the exit criterion, and thus increases the number of LP oracles per iteration. Therefore, it would be interesting to study how $\varphi_{\gamma,\theta,\lambda}$ impacts the number of LP calls per iteration. And finally, to compute a final complexity of how many LP calls ACondG method is needed to reach a desired accuracy.

It is worth noting that CondG method can also be used to design inexact versions of several projection methods, including but not limited to averaged projections method [33], Han's method [27] (see also [2]) or more generally Dykstra's alternating projection method [2]. In addition, other schemes such as block-iterations can be considered. For more variants of projections methods see [14, Section III]. For instance, one inexact version of averaged projection method for two sets is stated as follows:

Algorithm 4: Averaged projection method with inexact projections onto two sets

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Step 0. Let $(\lambda_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}$, and $(\theta_k)_{k\in\mathbb{N}}$ be sequences of nonnegative real numbers and $\varphi_k := \varphi_{\gamma_k,\theta_k,\lambda_k}$, as defined in (2). Let $x_0 \in A$, $y_0 \in B$, and set $z_0 := (x_0 + y_0)/2$. Initialize $k \leftarrow 0$.

Step 1. If $z^k \in A \cap B$, then **stop**.

Step 2. Using Agorithm 1, compute $\mathrm{CondG}_A\left(\varphi_k, x^k, z^k\right)$ and $\mathrm{CondG}_B\left(\varphi_k, y^k, z^k\right)$ and set the next iterate z^{k+1} as

$$z^{k+1} := \frac{1}{2}\left[\mathrm{CondG}_A\left(\varphi_k, x^k, z^k\right) + \mathrm{CondG}_B\left(\varphi_k, y^k, z^k\right)\right].$$

Step 3. Set $k \leftarrow k+1$, and go to **Step 1**.

Finally, it would also be interesting to extend the ACondG method to the convex feasibility problem with multiple involved sets or even for infinite families of sets in both finite and infinite dimensional spaces.

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