Acta Math. Hungar.
94 (4) (2002), 307–320.

CONTRIBUTIONS TO THE STUDY OF MONOTONE VECTOR FIELDS

J. X. DA CRUZ NETO (Teresina)[1], O. P. FERREIRA
and L. R. LUCAMBIO PÉREZ (Goiânia)

Abstract. We introduce the concept of a strongly monotone vector field on a Riemannian manifold and give an example. We also demonstrate relationships between different kinds of monotonicity of vector fields and different kinds of definiteness of its differential operator. Some topological and metric consequences of the strict and strongly monotone vector fields' existence are shown.

1. Introduction

A large class of non-convex constrained minimization problems can be seen as convex minimization problems in Riemannian manifolds. The study of the known optimization methods' extension to solve minimization problems on Riemannian manifolds is the subject of various works, see [3], [4], [5], [6], [13], [17] and their references.

A generalization of the convex minimization problem is the variational inequality problem. Several classes of monotone operators were introduced in the study of variational inequality problems and convergence properties of iterative methods to solve them.

The concept of monotonicity and strict monotonicity of the vector fields, that are defined on Riemannian manifolds are introduced in [10, 11]. We introduce the concept of a strongly monotone vector field, study its relation with strong convexity of function and give an example.

In Section 3, we establish the relationship between different classes of vector fields' monotonicity and their differential operators' definiteness. In Section 4, we prove that a strongly monotone vector field has only one singularity and that the square of such a vector field's norm is a coercive map.

In Section 5, we study some topological and metric consequences of the existence of strictly monotone vector fields, showing that if there exists a strictly monotone vector field then there exists no closed geodesic in this manifold. If, moreover, the manifold is not compact and has non-negative sectional curvature then its soul has dimension 0 and the manifold is therefore diffeomorphic to R^n. It will also be shown that if the manifold has non-

[1] Research of this author was partially supported by PRONEX, CNPq, Brazil.
Key words and phrases: monotone vector field, Riemannian manifold, convex function.
1991 Mathematics Subject Classification: 53A30, 53C22, 57S25, 58C20.

positive sectional curvature everywhere and there exists a strongly monotone vector field defined on it, then the volume of this manifold is infinite.

2. Basics concepts

In this section some frequently used notations, basic definitions and important properties of Riemannian manifolds are cited. They can be found in any introductory book on Riemannian geometry, for example [2] and [14]. Throughout this paper, all mentioned manifolds are smooth and connected and all functions and vector fields are smooth.

Let a manifold M be given and denote the space of vector fields over M by $\mathfrak{X}(M)$, the tangent space of M at p by T_pM and the ring of functions over M by $\mathfrak{F}(M)$. Let M be endowed with a Riemannian metric $\langle , \rangle$, with the corresponding norm denoted by $\| \; \|$, so that M is now a *Riemannian* manifold. Remember that the metric can be used to define the length of a piecewise smooth curve $c : [a,b] \to M$ joining p to q, p, $q \in M$, i.e. $c(a) = p$ and $c(b) = q$, by $l(c) = \int_a^b \left\| c'(t) \right\| \, dt$. Minimizing this length functional over the set of all such curves we obtain a distance $d(p,q)$ which induces the original topology on M. The metric induces a map $f \in \mathfrak{F}(M) \mapsto \operatorname{grad} f \in \mathfrak{X}(M)$ which associates to each f its *gradient* by the rule $\langle \operatorname{grad} f, X \rangle = df(X)$, $X \in \mathfrak{X}(M)$. The chain rule is generalized to this setting in the usual way: $(f \circ c)'(t) = \left\langle \operatorname{grad} f\bigl(c(t)\bigr), c'(t) \right\rangle$. In particular, if f assumes either a maximum or a minimum value at a point $p \in M$ then $\operatorname{grad} f(p) = 0$.

Let ∇ be the Levi-Civita connection associated to $\bigl(M, \langle , \rangle\bigr)$. If c is a curve joining points p and q in M, then, for each $t \in [a,b]$, ∇ induces an isometry, relative to $\langle , \rangle$, $P(c)_t^a : T_{c(a)}M \to T_{c(t)}M$, the so-called *parallel transport* along c from $c(a)$ to $c(t)$. The inverse map of $P(c)_t^a$ is denoted by $P(c^{-1})_t^a : T_{c(t)}M \to T_{c(a)}M$. A vector field V along c is said to be *parallel* if $\nabla_{c'} V = 0$. If c' itself is parallel we say that c is a *geodesic*. The geodesic equation $\nabla_{\gamma'} \gamma' = 0$ is a nonlinear ordinary differential equation of second order, and γ is determined by its position and velocity at one point. It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment.* A geodesic segment joining p to q in M is said to be *minimal* if its length equals $d(p,q)$.

A Riemannian manifold is *complete* if geodesics are defined for any values of t. Hopf-Rinow's theorem asserts that if this is the case then any pair of points, say p and q, in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M,d) is a complete metric space and bounded and closed subsets are compact. In this paper, all manifolds are assumed to be complete. The *exponential map* $\exp_p : T_pM \to M$ is defined by $\exp_x v$

$= \gamma_v(1,x)$, where $\gamma(\cdot) = \gamma_v(\cdot,p)$ is the geodesic defined by its position p and velocity v at p. We can prove that $\exp_p tv = \gamma_v(t,p)$ for any values of t.

The fundamental local invariant of Riemannian manifolds is the *curvature tensor* R defined for $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $[\,,\,]$ is the Lie bracket. Clearly, R is a tensor of type (3,1). Given $p \in M$ and a 2-dimensional subspace $\sigma \subset T_pM$, the quantity

$$K(u,v) = \frac{\langle R(u,v)v, u \rangle}{\|u\|^2 \|v\|^2 - \langle u, v\rangle^2}$$

does not depend on the basis $\{u,v\} \subset \sigma$. Hence, $K(u,v) = K(\sigma)$ depends only on σ and is called the *sectional curvature* of σ at x.

Some interesting results are obtained when the curvature's sign is constant. If $K(\sigma) \leqq 0$ for any σ, then we refer to the manifold as a *manifold with nonpositive curvature,* in the other case, we refer to it as a *manifold with nonnegative curvature.*

The next two important results are valid in manifolds with nonpositive sectional curvature.

THEOREM 2.1. *Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Then M is diffeomorphic to the Euclidean space R^n, $n = \dim M$. More precisely, at any point $p \in M$, the exponential mapping* $\exp_p : T_pM \to M$ *is a diffeomorphism.*

PROOF. See [2] and [14], p. 221, Theorem 4.1(2). $\square$

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold.* From now on H is a Hadamard manifold. Theorem 2.1 says that H has the same topology and differential structure as the Euclidean space. Furthermore, some geometrical properties of the Hadamard manifold are similar to some geometrical properties of the Euclidean space.

A *geodesic triangle* $\Delta(p_1p_2p_3)$ in H is the set consisting of three distinct points p_1, p_2, p_3 called the *vertices* and three geodesic segments γ_i joining p_{i+1} to p_{i+2} called the *sides,* where $i = 1,2,3 \pmod 3$.

THEOREM 2.2. *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle in the manifold H. Denote the geodesic segment joining p_{i+1} to p_{i+2} by γ_i and set $\ell_i = l(\gamma_i)$ and $\theta_i = \sphericalangle\big(\gamma'_{i-1}(0), -\gamma'_{i+1}(\ell_{i+1})\big)$, where $i = 1,2,3 \pmod 3$. Then*

$$\theta_1 + \theta_2 + \theta_3 \leqslant \pi, \tag{2.1}$$

$$\ell_{i+1}^2 + \ell_{i+2}^2 - 2\ell_{i+1}\ell_{i+2}\cos\theta_i \leqslant \ell_i^2 \tag{2.2}$$

and

$$\ell_i \cos\theta_{i+1} + \ell_{i+1}\cos\theta_i \geqslant \ell_{i+2}. \tag{2.3}$$

PROOF. Inequalities (2.1) and (2.2) are proved in [14] Proposition 4.5, p. 223. Inequality (2.3) is an immediate consequence of (2.2). □

A Riemannian submanifold S of M is called *totally geodesic* if all geodesics in S are also geodesics in M and it is said to be *totally convex* if for all points p, q in S, all geodesics joining p to q are contained in S. The following results are true when the sectional curvature of M is nonnegative everywhere. The first theorem is due to J. Cheeger and D. Gromoll.

THEOREM 2.3. *Let M be a complete noncompact Riemannian manifold of nonnegative curvature. Then M contains a compact totally geodesic submanifold S with $\dim S < \dim M$, which is totally convex. Furthermore, M is diffeomorphic to the normal bundle of S.*

PROOF. See [14], Theorem 3.4, p. 215. □

Beginning at any point of M such an S, called the *soul* of M, can be built. In [12], G. Perelman proved the following result.

THEOREM 2.4. *Let M be a complete non compact Riemannian manifold of nonnegative sectional curvature. If there exists a point of M at which the sectional curvature is positive, then the soul S of M consists of one point, which is called a simple point, and M is diffeomorphic to R^n.*

PROOF. See [12]. □

The *differential* of $X \in \mathfrak{X}(M)$ is the linear operator $A_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by $A_X(Y) := \nabla_Y X$. Then, to each point $p \in M$, we assign the linear map $A_X(p) : T_pM \to T_pM$ defined by

$$A_X(p)v = \nabla_v X. \tag{2.4}$$

If $X = \operatorname{grad} f$, where $f : M \to R$, then $A_X(p)$ is the *Hessian* of f at p and is denoted by $\operatorname{Hess} f_p$.

The *divergence* of $X \in \mathfrak{X}(M)$ is the trace of its differential. Therefore, for a fixed $p' \in M$, a neighborhood $\Omega \subset M$ of p' and a local orthonormal basis $E_1, E_2, \dots, E_n$, we get that

$$\operatorname{div} X = \operatorname{trace} A_X = \sum_{i=1}^{n} \langle A_X(E_i), E_i \rangle. \tag{2.5}$$

Let M be a Riemannian manifold of dimension n. Take an atlas $\mathcal{A} = \left\{ \left(\Omega_\Lambda, \mathbf{x}_\Lambda, x^i_\Lambda \right) \right\}$ and a partition of unity $\{\tau_\Lambda\}$ subordinate to Ω_Λ. For each function $f : M \to R$, define its integral as

$$\int_M f d\nu := \sum_\Lambda \int_{\mathbf{x}_\Lambda(\Omega_\Lambda)} \left(\tau_\Lambda \cdot f \cdot \sqrt{\det\left(g^\Lambda_{ij}\right)} \right) \circ \mathbf{x}_\Lambda^{-1} dx^1_\Lambda \dots dx^n_\Lambda,$$

where (g^Λ_{ij}) is the matrix of the metric of M with relation to local coordinates (x^i_Λ). We can prove that this definition does not depend on the choice of the atlas and the partition of unity. Now the *volume* of $\Omega \subseteqq M$ is given by

$$\operatorname{Vol}(\Omega) := \int_M \chi_\Omega d\nu, \tag{2.6}$$

where $\chi_\Omega : M \to R$ is defined by $\chi_\Omega(p) = 1$ if $p \in \Omega$ and $\chi_\Omega(p) = 0$ if $p \notin \Omega$.

3. Strong monotonicity

Let M be a Riemannian manifold, $X \in \mathfrak{X}(M)$ a vector field and γ a geodesic in M. Consider the real function $\varphi_{(X,\gamma)} : M \to R$ defined by

$$\varphi_{(X,\gamma)}(t) = \left\langle \gamma'(t), X\left(\gamma(t)\right) \right\rangle.$$

In [10, 11] S. Z. Németh defines a vector field X as *monotone* in M if $\varphi_{(X,\gamma)}$ is monotone (non-decreasing) for all geodesic γ in M and as *strictly monotone* if $\varphi_{(X,\gamma)}$ is strictly monotone. It is also natural to extend the concept of strong monotonicity to vector fields defined in Riemannian manifolds, and as far as we know that has not been done yet.

DEFINITION 3.1. The vector field $X \in \mathfrak{X}(M)$ is called strongly monotone when there exists $\lambda > 0$ such that, for any geodesic γ in M, the real function $\Psi_{(X,\gamma)}(t) = \varphi_{(X,\gamma)}(t) - \lambda \left\| \gamma'(0) \right\|^2 t$ is monotone.

When a reference to the vector field X is not necessary, or when it is implicit, we will use the notation Ψ_γ instead of $\Psi_{(X,\gamma)}$. Note that if α is a reparametrization of γ then Ψ_γ is monotone if and only if Ψ_α is monotone, too.

PROPOSITION 3.1. *Let M be Riemannian manifold.*

(i) *The vector field $X \in \mathfrak{X}(M)$ is monotone in M if and only if, for any two points p, $q \in M$ and each geodesic γ joining p to q, with $\gamma(0) = p$ and $\gamma(t) = q$,*

$$\left\langle \gamma'(0), P(\gamma^{-1})^0_t X(q) - X(p) \right\rangle \geqq 0. \tag{3.1}$$

If the inequality in (3.1) *is always strict, then X is strictly monotone.*

(ii) *The vector field $X \in \mathfrak{X}(M)$ is strongly monotone in M if and only if, for any two points $p, q \in M$ and each geodesic γ joining p to q, with $\gamma(0) = p$ and $\gamma(t) = q$, there exists $\lambda > 0$ such that*

$$\left\langle \gamma'(0), P(\gamma^{-1})_t^0 X(q) - X(p) \right\rangle \geqq \lambda \left\| \gamma'(0) \right\|^2 t. \tag{3.2}$$

Proof. We will prove only (ii). The proof of (i) is analogous.

Consider a geodesic γ and real numbers t_1, t_2. Set $p = \gamma(t_1)$ and $q = \gamma(t_2)$. Define the geodesic $\alpha(t) = \gamma\big(t_1 + t(t_2 - t_1)\big)$. Observe that $\alpha(0) = p$, $\alpha(1) = q$ and that $\alpha'(t) = (t_2 - t_1)\gamma'\big(t_1 + t(t_2 - t_1)\big)$. Then, by definition of Ψ_γ, it follows that

$$\begin{aligned}
&(t_2 - t_1)\big(\Psi_\gamma(t_2) - \Psi_\gamma(t_1)\big) \\
&= (t_2 - t_1)\Big(\big\langle \gamma'(t_2), X\big(\gamma(t_2)\big) \big\rangle - \big\langle \gamma'(t_1), X\big(\gamma(t_1)\big) \big\rangle - \lambda \big\| \gamma'(0) \big\|^2 (t_2 - t_1)\Big) \\
&= \big\langle \alpha'(1), X(q) \big\rangle - \big\langle \alpha'(0), X(p) \big\rangle - \lambda \big\| \alpha'(0) \big\|^2 \\
&= \big\langle P(\alpha^{-1})_1^0 \alpha'(1), P(\alpha^{-1})_1^0 X(q) \big\rangle - \big\langle \alpha'(0), X(p) \big\rangle - \lambda \big\| \alpha'(0) \big\|^2 \\
&= \big\langle \alpha'(0), P(\alpha^{-1})_1^0 X(q) - X(p) \big\rangle - \lambda \big\| \alpha'(0) \big\|^2.
\end{aligned}$$

Thus if $\big\langle \alpha'(0), P(\alpha^{-1})_1^0 X(q) - X(p) \big\rangle - \lambda \big\| \alpha'(0) \big\|^2 \geqq 0$, then $(t_2 - t_1)\big(\Psi_\gamma(t_2) - \Psi_\gamma(t_1)\big) \geqq 0$. Since the geodesic γ and the real numbers t_1, t_2 are arbitrary, the function Ψ_γ is monotone and, by Definition 3.1, it follows that X is strongly monotone.

Now, given $p, q \in M$, $p \neq q$, and a geodesic γ such that $\gamma(0) = p$, $\gamma(t) = q$, $t > 0$, define $\alpha(s) = \gamma(ts)$.

$$\begin{aligned}
&\big\langle \gamma'(0), P(\gamma^{-1})_t^0 X(q) - X(p) \big\rangle - \lambda \big\| \gamma'(0) \big\|^2 t \\
&= \frac{1}{t}\big(\big\langle P(\gamma^{-1})_t^0 t\gamma'(t), P(\gamma^{-1})_t^0 X(p) \big\rangle - \lambda \big\| \gamma'(0) \big\|^2 t^2 - \big\langle t\gamma'(0), X(p) \big\rangle\big) \\
&= \frac{1}{t}\Big(\big(\big\langle \alpha'(1), X(q) \big\rangle - \lambda \big\| \alpha'(0) \big\|^2\big) - \big\langle \alpha'(0), X(p) \big\rangle\Big) \\
&= \frac{1}{t}\Big(\big(\varphi_{(X,\alpha)}(1) - \lambda \big\| \alpha'(0) \big\|^2\big) - \varphi_{(X,\alpha)}(0)\Big) = \frac{1}{t}\big(\Psi_\alpha(1) - \Psi_\alpha(0)\big).
\end{aligned}$$

Hence, if X is strongly monotone then, by Definition 3.1, Ψ_γ is monotone, which implies that $\frac{1}{t}\big(\Psi_\alpha(1) - \Psi_\alpha(0)\big) \geqq 0$, and inequality (3.2) is in fact valid. □

REMARK 3.1. Observe that, when M is a Hadamard manifold, (3.2) can be written as $\langle \exp_p{}^{-1} q, P(\gamma^{-1})_1^0 X(q) - X(p) \rangle \geqq \lambda d^2(p,q)$, where γ is the geodesic joining p to q with $\gamma(0) = p$ and $\gamma(1) = q$.

Now we give an example of a strongly monotone vector field. Set $p' \in H$. By Hadamard's Theorem, the exponential map has inverse, $\exp_{p'}^{-1} : H \to T_{p'}H$, hence $d(p,p') = \| \exp_{p'}^{-1} p \|$. Therefore the function $\rho_{p'} : H \to R$, defined by

$$\rho_{p'}(p) = \frac{1}{2} d^2(p,p'), \tag{3.3}$$

is smooth and $\operatorname{grad} \rho_{p'}(p) = -\exp_p^{-1} p'$, see [14].

Now let the function f be strictly increasing with $f(0) = 0$. Using the Riemannian distance and the exponential map, the *f-position vector field,* introduced in [9] by S. Z. Németh, is defined by

$$P^f(p) = \begin{cases} \dfrac{f\big(d(p',p)\big)}{d(p',p)} \big(-\exp_p{}^{-1} p' \big), & \text{if } \ p \neq p' \\ 0, & \text{if } \ p = p' \end{cases} \tag{3.4}$$

or equivalently

$$P^f(p) = \begin{cases} \dfrac{f\big(d(p',p)\big)}{d(p',p)} \operatorname{grad} \rho_{p'} p, & \text{if } \ p \neq p' \\ 0, & \text{if } \ p = p'. \end{cases} \tag{3.5}$$

In [9] it is proved that P^f is strictly monotone. Note that $P^{\mathrm{id}}(p) = \operatorname{grad} \rho_{p'}(p)$, where id is the identity function, is strictly monotone. In the next proposition, it will be proved that $\operatorname{grad} \rho_{p'}(p)$ is strongly monotone.

PROPOSITION 3.2. *Let $\rho_{p'}$ be the function defined by* (3.3)*. For any fixed $p' \in H$, the gradient vector field* $\operatorname{grad} \rho_{p'}(p)$ *is strongly monotone.*

PROOF. Take a geodesic γ in H. By Proposition 3.1 it is enough to prove that $\Psi_{(X,\gamma)}$, where $X = \operatorname{grad} \rho_{p'}$ is monotone, i.e., $(t_2 - t_1)\big(\Psi_\gamma(t_2) - \Psi_\gamma(t_1)\big) \geqq 0$ for all $t_1, t_2 \in R$. We have two cases: γ goes through p' or γ does not go through p'.

Suppose that γ goes through p'. Then there exists t_0 such that $\gamma(t_0) = p'$. Without loss of generality, assume that $t \geqq t_0$. Then

$$\begin{aligned} \Psi_\gamma(t) &= \big\langle \gamma'(t), \operatorname{grad} \rho_{p'}\big(\gamma(t)\big) \big\rangle - \big\| \gamma'(0) \big\|^2 t \\ &= \big\langle \gamma'(t), -\exp_{\gamma(t)}^{-1} p' \big\rangle - \big\| \gamma'(0) \big\|^2 t \end{aligned}$$

$$= \big\| \gamma'(0) \big\|^2 (t - t_0) - \big\| \gamma'(0) \big\|^2 t = -\big\| \gamma'(0) \big\|^2 t_0.$$

Therefore Ψ_γ is monotone.

Now suppose that γ does not go through p'. With t_1 and t_2 given, consider the geodesic triangle $\Delta(p_0p_1p_2)$, where $p_0 = p'$, $p_1 = \gamma(t_1)$ and $p_2 = \gamma(t_2)$. Denote γ_0, γ_1 and γ_2 the normalized geodesic segments joining p_1 to p_2, p_2 to p_0, and p_0 to p_1, respectively. Set the numbers $\ell_0 = l(\gamma_0)$, $\ell_1 = l(\gamma_1)$, $\ell_2 = l(\gamma_2)$ and the angles $\theta_1 = \sphericalangle\big(\gamma_0'(0), -\gamma_2'(\ell_2)\big)$, $\theta_2 = \sphericalangle\big(\gamma_1'(0), -\gamma_0'(\ell_0)\big)$ and $\theta_0 = \sphericalangle\big(\gamma_2'(0), -\gamma_1'(\ell_1)\big)$. Note that $\ell_0 = (t_2 - t_1)\|\gamma'\|$, $\operatorname{grad}\rho_{p'}(p_1) = \ell_2\gamma_2'(\ell_2)$ and that $\operatorname{grad}\rho_{p'}(p_2) = -\ell_1\gamma_1'(0)$. Since $\Psi_\gamma(t_1) = \big\langle \gamma'(t_1), \ell_2\gamma_2'(\ell_2) \big\rangle - \|\gamma'\|^2 t_1$ and $\Psi_\gamma(t_2) = \big\langle \gamma'(t_2), -\ell_1\gamma_1'(0) \big\rangle - \|\gamma'\|^2 t_2$ we have that

$$(t_2 - t_1)\big(\Psi_\gamma(t_2) - \Psi_\gamma(t_1)\big)$$

$$= (t_2 - t_1)\Big(\big(\big\langle \gamma'(t_2), -\ell_1\gamma_1'(0) \big\rangle - \|\gamma'\|^2 t_2\big) - \big(\big\langle \gamma'(t_1), \ell_2\gamma_2'(\ell_2) \big\rangle - \|\gamma'\|^2 t_1\big)\Big)$$

$$= (t_2 - t_1)\big(\big\langle -\gamma'(t_2), \ell_1\gamma_1'(0) \big\rangle + \big\langle \gamma'(t_1), -\ell_2\gamma_2'(\ell_2) \big\rangle - (t_2 - t_1)\|\gamma'\|^2\big)$$

$$= \ell_0\ell_1 \cos\theta_2 + \ell_0\ell_2 \cos\theta_1 - \ell_0^2 = \ell_0(\ell_1\cos\theta_2 + \ell_2\cos\theta_1 - \ell_0).$$

Considering $i = 1$ in (2.3), we obtain that $\ell_1\cos\theta_2 + \ell_2\cos\theta_1 - \ell_0 \geqq 0$. Therefore Ψ_γ is monotone for each geodesic γ and the assertion of the theorem follows from Proposition 3.1. □

PROPOSITION 3.3. *Let be M a Riemannian manifold and let X be a vector field on M.*

(i) *X is monotone if and only if $\big\langle A_X(p)v, v \big\rangle \geqq 0$ for any $p \in M$ and $v \in T_pM$.*

(ii) *If $\big\langle A_X(p)v, v \big\rangle > 0$ for any $p \in M$ and $v \in T_pM$, then X is strictly monotone.*

(iii) *X is strongly monotone if and only if there exists $\lambda > 0$ such that $\big\langle A_X(p)v, v \big\rangle \geqq \lambda\|v\|^2$, for any $p \in M$ and $v \in T_pM$.*

PROOF. (i) and (ii) are proved in [10], Corollary 2.7, (i) and (ii). The vector field X is strongly monotone if and only if there exists $\lambda > 0$ such that for all geodesics γ the real function $\Psi_\gamma(t) = \varphi_{(X,\gamma)} - \lambda\|\gamma'\|^2 t$ is monotone. But the function Ψ_γ is monotone if and only if

$$\Psi_\gamma'(t) = \big\langle \gamma'(t), \nabla_{\gamma'(t)}X \big\rangle - \lambda\|\gamma'\|^2 = \big\langle \gamma'(t), A_X\big(\gamma(t)\big).\gamma'(t) \big\rangle - \lambda\|\gamma'\|^2 \geqq 0$$

for all t. Therefore the inequality $\big\langle \gamma'(t), A_X\big(\gamma(t)\big).\gamma'(t) \big\rangle - \lambda\|\gamma'\|^2 \geqq 0$ is valid for each geodesic γ if and only if $\big\langle A_X(p) \cdot v, v \big\rangle \geqq \lambda\|v\|^2$ for any $p \in M$ and all $v \in T_pM$. □

A function $f : M \to R$ is called *convex,* respectively *strictly convex* in M if the composition of f with each geodesic γ of M is a convex, respectively *strictly convex* real function. We will call f *strongly convex* if and only if its composition with each geodesic in M is strongly convex. In [8] several convexity notions were related to the corresponding monotonicity notions. Next we relate the notion of strong convexity to that of strong monotonicity.

PROPOSITION 3.4. *Let M be a Riemannian manifold and take $f : M \to R$.*

(i) *The function f is convex if and only if the vector field* $\operatorname{grad} f$ *is monotone.*

(ii) *The function f is strictly convex if and only if the vector field* $\operatorname{grad} f$ *is strictly monotone.*

(iii) *The function f is strongly convex if and only if the vector field* $\operatorname{grad} f$ *is strongly monotone.*

PROOF. For (i) and (ii) see [13] and [17]. By definition, f is strongly convex if and only if $\eta(t) = f\big(\gamma(t)\big)$ is strongly convex for each geodesic γ. By Proposition 1.1.2 of [7], p. 144, η is strongly convex if and only if there exists $\delta > 0$ such that $\zeta(t) = \eta(t) - \delta t^2$ is convex. But $\zeta(t)$ is convex if and only if $\zeta'(t)$ is monotone. Taking $\lambda \big\| \gamma'(0) \big\|^2 = 2\delta$ we get that $\Psi_{(\operatorname{grad} f, \gamma)}(t) = \zeta'(t)$ is monotone if and only if $\operatorname{grad} f$ is strongly monotone. Therefore, by Definition 3.1 it follows that f is strongly convex if and only if the vector field $\operatorname{grad} f$ is strongly monotone. □

COROLLARY 3.1. *Fix $p' \in H$. The map $\rho_{p'}(p)$ defined by* (3.3) *is strongly convex and $\big\langle \operatorname{Hess} \rho_{p'}(p) \cdot v, v \big\rangle \geqq \|v\|^2$ for any $p \in M$ and $v \in T_pM$.*

PROOF. By Proposition 3.2, the vector field $\operatorname{grad} \rho_{p'}(p)$ is strongly monotone. Then, by Proposition 3.4(iii), $\rho_{p'}(p)$ is strongly convex. Moreover, by Proposition 3.3(iii), there exists $\lambda > 0$ such that $\big\langle \operatorname{Hess} \rho_{p'}(p) \cdot v, v \big\rangle \geqq \|v\|^2$ for any $p \in M$. □

4. Some properties of strongly monotone vector fields

Next we study some properties of strongly monotone vector fields. Let M be a Riemannian manifold and $X \in \mathfrak{X}(M)$ a strongly monotone vector field. Consider the map $f : M \to R$ defined by

$$f(p) = \frac{1}{2} \big\| X(p) \big\|^2. \tag{4.1}$$

PROPOSITION 4.1. *If $X \in \mathfrak{X}(M)$ is strongly monotone then f is coercive, i.e., for any fixed p', $\lim_{d(p',p)\to\infty} f(p) = \infty$.*

PROOF. Suppose that there are $c > 0$ and a sequence $\{p_k\} \subset M$ such that $\lim_{k\to\infty} d(p', p_k) = \infty$ and $f(p_k) \leqq c$, for all k. Let γ_k be the normalized geodesic with $\gamma_k(0) = p'$ and $\gamma_k(t_k) = p_k$, $t_k > 0$. Then, by Proposition 3.1(ii), there exists $\lambda > 0$ such that

$$\lambda d(p', p_k) \leqq \lambda t_k \leqq \left\langle {\gamma_k}'(0), P(\gamma_k^{-1})_{t_k}^{0} X(p_k) - X(p') \right\rangle .$$

Using Cauchy–Schwarz inequality and knowing that $f(p_k) \leqq c$ for any k, we get in contradiction with our assumption that $\lambda d(p', p_k) \leqq \sqrt{2c} + \left\| X(p') \right\|$, i.e., $d(p', p_k)$ is bounded. □

COROLLARY 4.1. *If X is strongly monotone then there exists only one $\hat{p} \in M$ such that $X(\hat{p}) = 0$.*

PROOF. By Proposition 4.1 f is coercive. Therefore f has a minimum. Set $\hat{p}$ as a minimizer of f. Then

$$0 = df_{\hat{p}} \cdot v = \left\langle A_X(\hat{p}) \cdot v, X(\hat{p}) \right\rangle , \tag{4.2}$$

for all $v \in T_{\hat{p}}M$. Considering $v = X(\hat{p})$ and using Proposition 3.3(iii) we get that

$$0 = \left\langle A_X(\hat{p}).X(\hat{p}), X(\hat{p}) \right\rangle \geqq \lambda \left\| X(\hat{p}) \right\| \tag{4.3}$$

for some $\lambda > 0$. Then, by (4.3), $\left\| X(\hat{p}) \right\| = 0$. The uniqueness is an immediate consequence of the strong monotonicity's definition. □

PROPOSITION 4.2. *Let $X \in \mathfrak{X}(M)$ be a vector field and $\alpha : [0, \omega) \to M$ the integral curve of X through $p \in M$. If X is strongly monotone, then there exists $\lambda > 0$ such that $\left\| X\big(\alpha(t)\big) \right\| \geqq \left\| X(p) \right\| e^{\lambda t}$ for all $t \in [0, \omega)$.*

PROOF. If $\left\| X(p) \right\| = 0$, then $\alpha(t) = p$ for all $t \in [0, \omega)$ and the result is true. Suppose that $\left\| X(p) \right\| \neq 0$. Define $\psi(t) = f\big(\alpha(t)\big)$, for all $t \in [0, \omega)$. Then, by Proposition 3.3(iii), there exists $\lambda > 0$ such that

$$\psi'(t) = \left\langle A_X\big(\alpha(t)\big)\alpha'(t), X\big(\alpha(t)\big) \right\rangle \tag{4.4}$$

$$= \left\langle A_X\big(\alpha(t)\big) X\big(\alpha(t)\big), X\big(\alpha(t)\big) \right\rangle \geqq \lambda \left\| X\big(\alpha(t)\big) \right\|^2 = 2\lambda\psi(t) \geqq 0.$$

Hence ψ is non-decreasing. Since $\left\| X(p) \right\| \neq 0$, statement (4.4) implies that ψ is strictly increasing and positive. Furthermore, $\psi(t) \geqq \psi(0)e^{\lambda t}$ for all $0 \leqq t < \omega$, which implies the statement of the proposition. □

PROPOSITION 4.3. *Let $X \in \mathfrak{X}(M)$ be a vector field and $\alpha : (\omega, 0] \to M$ the integral curve of X through $p \in M$. If X is strongly monotone, then there exists $\lambda > 0$ such that $\|X(\alpha(t))\| \leqq \|X(p)\| e^{\lambda t}$ for all $t \in (\omega, 0]$.*

PROOF. If $\|X(p)\| = 0$, then $\alpha(t) = p$ for all $t \in [0, -\omega)$ and the result is true. Suppose that $\|X(p)\| \neq 0$. Define the vector field Y by $Y(q) = -X(q)$ and the functions $\beta : [0, -\omega) \to M$ by $\beta(t) = \alpha(-t)$, and $\phi : [0, -\omega) \to R$ by $\phi(t) = \frac{1}{2}\|Y(\beta(t)\|^2$. Then, by Proposition 3.3(iii), there exists $\lambda > 0$ such that

$$\text{(4.5)} \qquad \phi'(t) = \left\langle A_Y \beta'(t), Y(\beta(t)) \right\rangle = \left\langle A_Y Y(\beta(t)), Y(\beta(t)) \right\rangle$$

$$= -\left\langle A_X X(\beta(t)), X(\beta(t)) \right\rangle \leqq -\lambda \left\| X(\beta(t)) \right\|^2 \leqq -\lambda 2\phi(t).$$

Since $X(p) \neq 0$, this implies that $\phi(t) \leqq \phi(0)e^{-2\lambda t}$ for all $0 \leqq t < -\omega$, which yields the statement of the proposition. $\square$

5. Consequences of the existence of monotone vector fields

It is well known that the existence of convex functions imposes some topological consequences on the Riemannian manifold M, see [15]. The concept of monotonicity is, in a certain way, a generalization of the concept of convexity. Then a natural and logical sequence is that the existence of a monotone vector field on M imposes topological consequences also on M.

If γ is a closed geodesic then $\varphi_{(X,\gamma)}$ is constant, see [9]. Therefore, if M has a closed geodesic then all convex functions defined on M are trivial. In the next proposition we will prove a more general result.

PROPOSITION 5.1. *Let M be a complete Riemannian manifold. If there exists a strictly monotone $X \in \mathfrak{X}(M)$, then all totally geodesic compact submanifolds of M are trivial, i.e. they consist of simple points.*

PROOF. We derive a contradiction on assuming that there exists a nontrivial totally geodesic compact submanifold N of M. By Theorem 3.5 on p. 299 of [14] the submanifold N has a closed geodesic γ, thus $\varphi_{(X,\gamma)}(t)$ is constant and X is not strictly monotone. $\square$

PROPOSITION 5.2. *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. If there exists a strictly monotone $X \in \mathfrak{X}(M)$, then the soul S of M consists of one point and M is diffeomorphic to R^n.*

PROOF. Take $p \in M$ and build the soul S starting from p. By Theorem 2.3, the soul S is a totally geodesic compact submanifold of M. Then, by Proposition 5.1 the submanifold S consists of one point. Again, according to Theorem 2.3, M is diffeomorphic to the normal bundle of S. Since S is a simple point, it follows that the normal bundle of S is diffeomorphic to R^n. Therefore M is diffeomorphic to R^n. $\square$

REMARK 5.1. Substituting the hypothesis of an existing point where the sectional curvature is positive by the hypothesis of the existence of a strictly monotone vector field, we obtain in Proposition 5.2 the same result as in Theorem 2.4.

It is known that the existence of nontrivial convex functions on M has metric consequences; for example: the volume of M, as defined in (2.6), is unbounded. Next we will show that the existence of strongly monotone vector fields also imposes this metric consequence.

First we introduce some definitions. Given $X \in \mathfrak{X}(M)$, define the sets $M^0 = \{p \in M : X(p) \neq 0\}$ and ∂M^0 as the boundary of M^0. Note that M^0 is open. Define the vector field U on M^0 by

$$U(p) = \frac{X(p)}{\|X(p)\|}. \tag{5.1}$$

Fix $p \in M^0$ and a neighborhood Ω of p in M^0. Take the vector fields $E_1 = U$, $E_2, \dots, E_n$ building an orthonormal basis of T_qM for all $q \in \Omega$. Therefore, after some algebraic manipulations we get from (2.5) and (5.1) that

$$\operatorname{div} U = \sum_{i=2}^{n} \frac{1}{\|X\|} \langle A_X(E_i), E_i \rangle. \tag{5.2}$$

From Proposition 3.3(iii) and (5.2) it follows that if X is strongly monotone then there exists $\lambda > 0$ such that

$$\operatorname{div} U(q) \geqq \frac{(n-1)\lambda}{\|X(q)\|} \tag{5.3}$$

for all $q \in \Omega$.

THEOREM 5.1. *Let M be a complete Riemannian manifold. If $X \in \mathfrak{X}(M)$ is strongly monotone then the volume of M is infinite.*

PROOF. We derive a contradiction by assuming that M has finite volume. Let f be defined by (4.1). Since X is strongly monotone by Proposition 4.1, there exists $c > 0$ such that $M^c = \{p \in M : f(p) \geqq c\} \subset M^0$ is not empty. Let Φ_t be the flow generated by the vector field U, where U is the vector field by definition in (5.1). First we show for all $t > 0$ that $\Phi_t(M^c) \subset M^c$. Take

$p \in M^c$ and $\beta : [0,\omega) \to M$ as the integral curve of U through p. Define the function $\Upsilon(t) = f\big(\beta(t)\big)$. Note that

$$\Upsilon'(t) = \frac{1}{2}\frac{d}{dt}\big\langle X\big(\beta(t)\big), X\big(\beta(t)\big)\big\rangle = \big\langle \nabla_{\beta'(t)} X\big(\beta(t)\big), X\big(\beta(t)\big)\big\rangle$$

$$= \big\langle A_X \beta'(t), X\big(\beta(t)\big)\big\rangle = \left\langle A_X \frac{X\big(\beta(t)\big)}{\big\| X\big(\beta(t)\big)\big\|}, X\big(\beta(t)\big)\right\rangle \geqq \lambda \big\| X\big(\beta(t)\big)\big\|$$

for some $\lambda > 0$. Then $\Upsilon'(t) > 0$, i.e. Υ is a strictly increasing function. Therefore β is well defined and $\beta(t) \in M^c$ and $\Phi_t(M^c) \subset M^c$ for all $t > 0$. Since $\Phi_t(M^c) \subset M^c$ for all $t > 0$ then

$$\operatorname{Vol}\big(\Phi_t(M^c)\big) \leqq \operatorname{Vol}(M^c), \quad \forall\, t \geqq 0. \tag{5.4}$$

From [14], Lemma 5.12, p. 71, it follows that

$$\frac{d}{dt}\operatorname{Vol}\big(\Phi_t(M^c)\big)\bigg|_{t=0} = \int_{M^c} \operatorname{div} U\, d\nu. \tag{5.5}$$

From (5.3) and (5.5) it follows that $\frac{d}{dt}\operatorname{Vol}\big(\Phi_t(M^c)\big)\big|_{t=0} > 0$, implying that there exists $\varepsilon > 0$ such that $\operatorname{Vol}\big(\Phi_t(M^c)\big) > \operatorname{Vol}(M^c)$ for all $0 < t < \varepsilon$, in contradiction with (5.4). Thus M has infinite volume. $\square$

6. Final remarks

If M has a nonnegative sectional curvature everywhere and if a strictly monotone X exists, then M is diffeomorphic to R^n. If M has a nonpositive sectional curvature everywhere, i.e. it is a Hadamard manifold, then the map $\rho_{p'}$ is strongly convex for all $p' \in M$. Hence $\operatorname{grad}\rho_{p'}$ is strongly monotone. Therefore all Hadamard manifolds have infinite volume. In [1] it is proved that if there exists strictly convex $f : M \to R$, then M has infinite volume. The existence of strictly convex functions implies the existence of strictly monotone vector fields because, in that case, $\operatorname{grad} f$ is strictly monotone. It is clear that all strongly monotone vector fields are also strictly monotone. Then the result of [1] could be stronger or weaker than our result, depending on the answer to the following question: is there a Riemannian manifold M with $X \in \mathfrak{X}(M)$ being strictly monotone and without $f : M \to R$ being strictly convex?

References

[1] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.,* **145** (1969), 1–49.

[2] M. P. do Carmo, *Riemannian Geometry,* Birkhäuser (Boston, 1992).

[3] J. X. da Cruz Neto, L. L. Lima, and P. R. Oliveira, Geodesic algorithm in Riemannian manifolds, *Balkan Journal of Geometry its Applications,* **3** (1998), 89–100.

[4] J. X. da Cruz Neto, O. P. Ferreira and L. R. Lucambio Perez, A proximal regularization of the steepest descent method in Riemannian manifold, *Balkan Journal of Geometry its Applications,* **4** (1999), 1–8.

[5] O. P. Ferreira and P.R. Oliveira, Subgradient algorithm on Riemannian manifolds, *Journal of Optimization Theory and Applications,* **97** (1998), 93–104.

[6] D. Gabay, Minimizing a differentiable function over a differentiable manifolds, *Journal of Optimization Theory and Applications,* **37** (1982), 177–219.

[7] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms,* I and II, Springer-Verlag (1993).

[8] S. Z. Németh, Five kinds of monotone vector fields, *Pure and Applied Mathematics,* **9** (1998), 417–428.

[9] S. Z. Németh, Geodesic monotone vector fields, *Lobachevskii Journal of Mathematics,* **5** (1999), 13–28.

[10] S. Z. Németh, Monotonicity of the complementary vector field of a nonexpansive map, *Acta Math. Hungar.,* **84** (1999), 189–197.

[11] S. Z. Németh, Monotone vector fields, *Publ. Math. Debrecen,* **54** (1999), 437–449.

[12] G. Perelman, Proof of the soul conjecture of Cheeger and Gromoll, *J. Differential Geometry,* **40** (1994), 209–212.

[13] T. Rapcsák, *Smooth Nonlinear Optimization in R^n,* Kluwer Academic Publishers (Dordrecht, 1997).

[14] T. Sakai, *Riemannian Geometry,* Translations of Mathematical Monographs, Vol. **149**, American Mathematical Society (Providence, R.I., 1996).

[15] K. Shiohama, Topology of complete noncompact manifolds, *Geometry of Geodesics and Related Topics, Advanced Studies in Pure Mathematics,* **3** (1984), 423–450.

[16] C. Udriste, Convex functions on Riemannian manifolds, *St. Cerc. Mat.,* **28** (1976), 735–745.

[17] C. Udriste, *Convex functions and optimization methods on Riemannian manifolds,* Mathematics and its Applications, Vol. **297**, Kluwer Academic Publishers (1994).

(Received May 4, 2000; revised November 17, 2000)

DM/CCN/UNIVERSIDADE FEDERAL DO PIAUÍ
CAMPUS DA ININGA, 64.049-550
TERESINA, PI
BRAZIL
E-MAIL: JXAVIER@UFPI.BR

IME/UNIVERSIDADE FEDERAL DE GOIÁS
CAMPUS SAMAMBAIA
CAIXA POSTAL 131
CEP 74001-970
GOIÂNIA, GO
BRAZIL
E-MAIL: ORIZON@MAT.UFG.BR
LRLP@MAT.UFG.BR