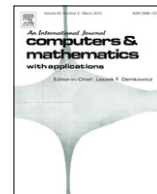




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Local convergence of the Gauss–Newton method for injective-overdetermined systems of equations under a majorant condition

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ABSTRACT

A local convergence analysis of the Gauss–Newton method for solving injective-overdetermined systems of nonlinear equations under a majorant condition is provided. The convergence as well as results on its rate are established without a convexity hypothesis on the derivative of the majorant function. The optimal convergence radius, the biggest range for uniqueness of the solution along with some other special cases are also obtained.

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1. Introduction

Let \mathbb{X} and \mathbb{Y} be real or complex Hilbert spaces. Let $\Omega \subseteq \mathbb{X}$ be an open set, and $F : \Omega \rightarrow \mathbb{Y}$ a continuously differentiable nonlinear function. Consider the *systems of nonlinear equations*

$$F(x) = 0. \quad (1)$$

If $F'(x)$ is invertible, the Newton method and its variants (see [1–4]) are the most efficient methods known for solving such systems. However, $F'(x)$ may not even be a square matrix. One simple example arises when $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{Y} = \mathbb{R}^m$, with $n \neq m$. In this case, $F'(x)$ is not invertible and (1) becomes an overdetermined system ($n < m$) or an underdetermined system ($n > m$). In general, if $F'(x)$ is injective or surjective, we say (1) is an injective-overdetermined or surjective-underdetermined system of equations, respectively.

If $F'(x)$ is not necessarily invertible, a generalized Newton method called the Gauss–Newton method can be used (see [5,6]). It is defined by

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

where $F'(x_k)^\dagger$ denotes the Moore–Penrose inverse of the linear operator $F'(x_k)$. This algorithm finds least-squares solutions of (1). These least-squares solutions, which may or may not be solutions of the original problem (1), are related to the nonlinear least squares problem

$$\min_{x \in \Omega} \|F(x)\|^2,$$

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that is, they are stationary points of $H(x) = \|F(x)\|^2$. This paper is focused on the case in which the least-squares solutions of (1) also solve (1). In the theory of nonlinear least squares problems, this case is called the zero-residual case.

Regarding the local and semi-local convergence analysis of the Newton and Gauss–Newton methods, in the last years there has been much work attempting to alleviate the assumption of Lipschitz continuity on the operator F' , see for example [1–15]. The main conditions that relax the Lipschitz continuity on the derivative is the majorant condition, used for example in [1–6], and the generalized Lipschitz condition according to X. Wang, used for example in [7–10,12–15]. In fact, as proved in [2], if the majorant function has convex derivative, these conditions are equivalent. Otherwise, the Wang’s condition can be seen as a particular case of the majorant condition. Moreover, the majorant formulation provides a clear relationship between the majorant function and the nonlinear operator under consideration, simplifying the proof of convergence substantially.

Our aim in this paper is to present a new local convergence analysis of the Gauss–Newton method for solving injective-overdetermined systems of equations under a majorant condition. The convergence, uniqueness, superlinear rate and an estimate of the best possible convergence radius will be established without a convexity hypothesis on the derivative of the majorant function, which was assumed in [5]. In addition to the special cases obtained in [5], the lack of convexity of the derivative of the majorant function in this analysis, allows us to obtain two new important special cases, namely, the convergence can be ensured under Hölder-like and generalized Lipschitz conditions. In the latter case, the results are obtained without assuming that the function that defines the condition is nondecreasing, thus generalizing Corollary 6.3 in [8]. Moreover, it is worth to mention that, similar to the convergence analysis of the Newton method (see [2]), the hypothesis of convex derivative of the majorant function or nondecreasing of the function which defines the generalized Lipschitz condition, are needed only to obtain quadratic convergence rate.

The organization of the paper is as follows. In Section 1.1, we list some notations and one basic result used in our presentation. In Section 2, we state the main result and in Section 2.1 some properties of the majorant function are established and the main relationships between the majorant function and the nonlinear function F are presented. The optimal ball of convergence and the uniqueness of the solution are also discussed in Section 2.1. In Section 2.2 our main result is proven and some applications of this result are obtained in Section 3. Some final remarks are offered in Section 4

1.1. Notation and auxiliary results

The following notations and results are used throughout our presentation. Let \mathbb{X} and \mathbb{Y} be Hilbert spaces. The open and closed ball at $a \in \mathbb{X}$ with radius $\delta > 0$ are denoted, respectively by

$$B(a, \delta) := \{x \in \mathbb{X}; \|x - a\| < \delta\}, \quad B[a, \delta] := \{x \in \mathbb{X}; \|x - a\| \leq \delta\}.$$

The set $\Omega \subseteq \mathbb{X}$ is an open set, the function $F : \Omega \rightarrow \mathbb{Y}$ is continuously differentiable, and $F'(x)$ has a closed image in Ω .

Some properties related to the Moore–Penrose inverse will be needed. More details about the Moore–Penrose inverse can be found in [16,17].

Let $A : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous and injective linear operator with closed image. The Moore–Penrose inverse $A^\dagger : \mathbb{Y} \rightarrow \mathbb{X}$ of A is defined by

$$A^\dagger := (A^*A)^{-1}A^*,$$

where A^* denotes the adjoint of the linear operator A .

Lemma 1. *Let $A, B : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous linear operator with closed image. If A is injective and $\|A^\dagger\| \|A - B\| < 1$, then B is injective and*

$$\|B^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \|A - B\|}.$$

2. Local analysis for the Gauss–Newton method

Our goal is to state and prove a local theorem for the Gauss–Newton method, which generalizes the Corollary 8 of [5], as well as Theorem 2 of [2]. First, we prove some results regarding the scalar majorant function, which relaxes the Lipschitz condition. Then, we establish the main relationships between the majorant function and the nonlinear function F . We also obtain the optimal ball of convergence and the uniqueness of the solution in a suitable region. Finally, we show well definedness and convergence, along with results on the convergence rates. The statement of the theorem is:

Theorem 2. *Let \mathbb{X} and \mathbb{Y} be Hilbert spaces, $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \rightarrow \mathbb{Y}$ be a continuously differentiable function such that F' has a closed image in Ω . Let $x_* \in \Omega$, $R > 0$, $\beta := \|F'(x_*)^\dagger\|$ and $\kappa := \sup \{t \in [0, R) : B(x_*, t) \subset \Omega\}$. Suppose that $F(x_*) = 0$, $F'(x_*)$ is injective and there exists a $f : [0, R) \rightarrow \mathbb{R}$ continuously differentiable such that*

$$\beta \|F'(x) - F'(x_* + \tau(x - x_*))\| \leq f'(\|x - x_*\|) - f'(\tau\|x - x_*\|), \tag{2}$$

for all $\tau \in [0, 1]$, $x \in B(x_*, \kappa)$ and

- (h1) $f(0) = 0$ and $f'(0) = -1$;
- (h2) f' is strictly increasing.

Let $\nu := \sup \{t \in [0, R) : f(t) < 0\}$, $\rho := \sup \{\delta \in (0, \nu) : [f(t)/f'(t) - t]/t < 1, t \in (0, \delta)\}$ and $r := \min \{\kappa, \rho\}$

Then, the sequences $\{x_k\}$ and $\{t_k\}$, with starting points $x_0 \in B(x_*, r)/\{x_*\}$ and $t_0 = \|x_0 - x_*\|$, respectively, such that

$$x_{k+1} = x_k - F'(x_k)^{\dagger}F(x_k), \quad t_{k+1} = |t_k - f(t_k)/f'(t_k)|, \quad k = 0, 1, \dots, \tag{3}$$

are well defined; $\{t_k\}$ is strictly decreasing, contained in $(0, r)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_*, r)$, it converges to the point x_* , which is the unique zero of F in $B(x_*, \sigma)$, where $\sigma := \sup\{t \in (0, \kappa) : f(t) < 0\}$, and there hold:

$$\lim_{k \rightarrow \infty} [\|x_{k+1} - x_*\|/\|x_k - x_*\|] = 0, \quad \lim_{k \rightarrow \infty} [t_{k+1}/t_k] = 0. \tag{4}$$

Moreover, if $f(\rho)/(\rho f'(\rho)) - 1 = 1$ and $\rho < \kappa$, then $r = \rho$ is the best possible convergence radius. If, additionally, given $0 \leq p \leq 1$

- (h3) the function $(0, \nu) \ni t \mapsto [f(t)/f'(t) - t]/t^{p+1}$ is strictly increasing,

then the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing and we have:

$$\|x_{k+1} - x_*\| \leq [t_{k+1}/t_k^{p+1}] \|x_k - x_*\|^{p+1} \leq [t_1/t_0^{p+1}] \|x_k - x_*\|^{p+1}, \quad k = 0, 1, \dots \tag{5}$$

Consequently, for $k \geq 0$,

$$\|x_k - x_*\| \leq \begin{cases} t_0 [t_1/t_0]^k, & \text{if } p = 0; \\ t_0 [t_1/t_0]^{((p+1)k-1)/p}, & \text{if } p \neq 0. \end{cases}$$

Remark 1. If $F'(x_*)$ is invertible in **Theorem 2** we obtain the local convergence of the Newton method for systems of nonlinear equations, as obtained in **Theorem 2** of [2].

Remark 2. In particular, if f' is convex, we can prove that (h3) holds with $p = 1$ and, therefore, in this case, we are led to the result proven in **Corollary 8** of [5], i.e., the local convergence of the Gauss–Newton method for solving the injective-overdetermined systems of equations. Hence, the additional assumption that the majorant function, f , has convex derivative, is only necessary in order to obtain quadratic convergence rate. This behavior is similar for the Newton method, see [2].

Example 1 (See [2]). The following continuously differentiable functions satisfy (h1)–(h3):

- (i) $f : [0, +\infty) \rightarrow \mathbb{R}$ such that $f(t) = t^{1+p} - t$;
- (ii) $f : [0, +\infty) \rightarrow \mathbb{R}$ such that $f(t) = e^{-t} + t^2 - 1$.

If $0 < p < 1$, the derivatives of both functions are not convex.

From now on, we assume that all the assumptions of **Theorem 2** hold, with the exception of (h3), which will be considered to hold only when explicitly stated.

2.1. Preliminary results

In this section, we will prove all the statements in **Theorem 2** regarding the sequence $\{t_k\}$ associated to the majorant function. The main relationships between the majorant function and the nonlinear operator will be also established, as well as the results in **Theorem 2** related to the uniqueness of the solution and the optimal convergence radius.

2.1.1. The scalar sequence

In this part, we will check the statements in **Theorem 2** involving $\{t_k\}$.

First of all, it is easy to see that the hypotheses (h1)–(h3) in **Theorem 2** coincide with those ones used in **Theorem 2** of [2]. Moreover, the constants κ , ν , ρ and σ also coincide. Hence, the proofs in this section, which can be found in **Section 2.1.1** of [2], will be omitted.

Proposition 3. The constants κ , ν and σ are positive and $t - f(t)/f'(t) < 0$, for all $t \in (0, \nu)$.

Proof. The proof follows as the one of **Proposition 3** of [2]. \square

According to (h2) and the definition of ν , we have $f'(t) < 0$ for all $t \in [0, \nu)$. Therefore, the Newton iteration map for f is well defined in $[0, \nu)$. Let us call it n_f :

$$n_f : [0, \nu) \rightarrow (-\infty, 0] \\ t \mapsto t - f(t)/f'(t). \tag{6}$$

Proposition 4. $\lim_{t \rightarrow 0} |n_f(t)|/t = 0$. As a consequence, $\rho > 0$ and $|n_f(t)| < t$ for all $t \in (0, \rho)$.

Proof. See the proof of Proposition 4 of [2]. \square

Using (6), it is easy to see that the sequence $\{t_k\}$ is equivalently defined as

$$t_0 = \|x_0 - x_*\|, \quad t_{k+1} = |n_f(t_k)|, \quad k = 0, 1, \dots \tag{7}$$

Corollary 5. The sequence $\{t_k\}$ is well defined, strictly decreasing and contained in $(0, \rho)$. Moreover, $\{t_k\}$ converges to 0 with superlinear rate, i.e., $\lim_{k \rightarrow \infty} t_{k+1}/t_k = 0$. If, additionally, (h3) holds, the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing.

Proof. The proof follows the same ideas of the proof of Corollary 5 of [2]. \square

2.1.2. Relationship of the majorant function with the nonlinear function

In this part we present the main relationships between the majorant function, f , and the nonlinear function, F .

Lemma 6. If $\|x - x_*\| < \min\{\nu, \kappa\}$, then $F'(x)^*F'(x)$ is invertible and

$$\|F'(x)^\dagger\| \leq \beta / |f'(\|x - x_*\|)|.$$

In particular, $F'(x)^*F'(x)$ is invertible in $B(x_*, r)$.

Proof. As $\|x - x_*\| < \min\{\nu, \kappa\}$, we have $f'(\|x - x_*\|) < 0$. Hence, using the definition of β , the inequality (2) and (h1), we have

$$\|F'(x_*)^\dagger\| \|F'(x) - F'(x_*)\| = \beta \|F'(x) - F'(x_*)\| \leq f'(\|x - x_*\|) - f'(0) < 1. \tag{8}$$

Since $F'(x_*)$ is injective, (8) implies, in view of Lemma 1, that $F'(x)$ is injective. So, $F'(x)^*F'(x)$ is invertible and, by the definition of r , we obtain that $F'(x)^*F'(x)$ is invertible for all $x \in B(x_*, r)$. Moreover, from Lemma 1 we also have

$$\|F'(x)^\dagger\| \leq \frac{\beta}{1 - \beta \|F'(x) - F'(x_*)\|} \leq \frac{\beta}{1 - (f'(\|x - x_*\|) - f'(0))} = \frac{\beta}{|f'(\|x - x_*\|)|},$$

where $f'(0) = -1$ and $f' < 0$ in $[0, \nu)$ are used for obtaining the last equality. \square

Now, it is convenient to study the linearization error of F at point in Ω . For this we define

$$E_F(x, y) := F(y) - [F(x) + F'(x)(y - x)], \quad y, x \in \Omega. \tag{9}$$

We will bound this error by the error in the linearization of the majorant function f

$$e_f(t, u) := f(u) - [f(t) + f'(t)(u - t)], \quad t, u \in [0, R). \tag{10}$$

Lemma 7. If $\|x - x_*\| < \kappa$, then $\beta \|E_F(x, x_*)\| \leq e_f(\|x - x_*\|, 0)$.

Proof. Since $B(x_*, \kappa)$ is convex, we obtain that $x_* + \tau(x - x_*) \in B(x_*, \kappa)$, for $0 \leq \tau \leq 1$. Thus, as F is continuously differentiable in Ω , the definition of E_F and some simple manipulations yield

$$\beta \|E_F(x, x_*)\| \leq \int_0^1 \beta \| [F'(x) - F'(x_* + \tau(x - x_*))] \| \|x_* - x\| \, d\tau.$$

From the last inequality and assumption (2), we obtain

$$\beta \|E_F(x, x_*)\| \leq \int_0^1 [f'(\|x - x_*\|) - f'(\tau\|x - x_*\|)] \|x - x_*\| \, d\tau.$$

Evaluating the above integral and using the definition of e_f , the statement follows. \square

In particular, Lemma 6 guarantees that $F'(x)^*F'(x)$ is invertible in $B(x_*, r)$ and, consequently, the Gauss–Newton iteration map is well defined. Let G_F be, the Gauss–Newton iteration map for F in that region:

$$G_F : B(x_*, r) \rightarrow \mathbb{Y} \\ x \mapsto x - F'(x)^\dagger F(x). \tag{11}$$

In the next proposition, we will establish an important relationship between the maps n_f and G_F . Consequently, we obtain that $B(x_*, r)$ is invariant under G_F . This result will be very important to ensure the good definition of the Gauss–Newton method.

Lemma 8. *If $\|x - x_*\| < r$, then $\|G_F(x) - x_*\| \leq |n_f(\|x - x_*\||)$. Consequently,*

$$G_F(B(x_*, r)) \subset B(x_*, r).$$

Proof. The first inequality is trivial for $x = x_*$, since $F'(x_*)^\dagger F(x_*) = 0$. Now, assume that $0 < \|x - x_*\| < r$. Lemma 6 implies that $F'(x)^*F'(x)$ is invertible. Hence, using $F(x_*) = 0$, some algebraic manipulation and (11), the following holds

$$G_F(x) - x_* = F'(x)^\dagger [F'(x)(x - x_*) - F(x) + F(x_*)].$$

From the last inequality, (9) and Lemmas 6 and 7, we obtain

$$\|G_F(x) - x_*\| \leq \|F'(x)^\dagger\| \|E_F(x, x_*)\| \leq \beta \|E_F(x, x_*)\| / |f'(\|x - x_*\||) \leq e_f(\|x - x_*\|, 0) / |f'(\|x - x_*\||).$$

On the other hand, taking into account that $f(0) = 0$, the definitions of e_f and n_f imply that

$$e_f(\|x - x_*\|, 0) / |f'(\|x - x_*\||) = |n_f(\|x - x_*\||).$$

Hence, the first statement follows by combining the last two inequalities.

For the second assertion, take $x \in B(x_*, r)$. Since $0 < \|x - x_*\| < r \leq \rho$, the first inequality of the lemma and the last inequality of Proposition 4 imply that $\|G_F(x) - x_*\| \leq |n_f(\|x - x_*\||) < \|x - x_*\|$, thus leading to the desired result. \square

Lemma 9. *If (h3) holds and $\|x - x_*\| \leq t < r$, then $\|G_F(x) - x_*\| \leq [|n_f(t)|/t^{p+1}] \|x - x_*\|^{p+1}$.*

Proof. The inequality is trivial for $x = x_*$. If $0 < \|x - x_*\| \leq t < r$, combining the assumption (h3) and (6), we obtain $|n_f(\|x - x_*\||) / \|x - x_*\|^{p+1} \leq |n_f(t)|/t^{p+1}$. So, using Lemma 8, the statement follows. \square

2.1.3. Optimal ball of convergence and uniqueness

In this section, we obtain the optimal convergence radius and the uniqueness of the solution.

Lemma 10. *If $f(\rho)/(\rho f'(\rho)) - 1 = 1$ and $\rho < \kappa$, then $r = \rho$ is the optimal convergence radius.*

Proof. Assume that $f(\rho)/(\rho f'(\rho)) - 1 = 1$ and $\rho < \kappa$. Define the function $h : (-\kappa, \kappa) \rightarrow \mathbb{R}$ by

$$h(t) = \begin{cases} -f(-t), & t \in (-\kappa, 0], \\ f(t), & t \in [0, \kappa). \end{cases} \tag{12}$$

It is straightforward to show that $h(0) = 0$, $h'(0) = -1$, $h'(t) = f'(|t|)$ and

$$|h'(0)| |h'(t) - h'(\tau t)| \leq f'(|t|) - f'(\tau|t|), \quad \tau \in [0, 1], \quad t \in (-\kappa, \kappa).$$

So, for $F = h$, $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $\Omega = (-\kappa, \kappa)$ and $x_* = 0$ the assumptions of Theorem 2 are satisfied. Thus, as $\rho < \kappa$, it suffices to show that the Gauss–Newton method applied for solving $h(t) = 0$, with starting point $x_0 = -\rho$, does not converge. As $f(\rho)/(\rho f'(\rho)) - 1 = 1$, the definition of h in (12) yields

$$x_1 = -\rho - h(-\rho)/h'(-\rho) = -\rho + f(\rho)/f'(\rho) = [f(\rho)/(\rho f'(\rho)) - 1]\rho = \rho.$$

Again, the definition of h in (12) and the assumption $f(\rho)/(\rho f'(\rho)) - 1 = 1$ lead to

$$x_2 = \rho - h(\rho)/h'(\rho) = \rho - f(\rho)/f'(\rho) = -[f(\rho)/(\rho f'(\rho)) - 1]\rho = -\rho.$$

Therefore, the Gauss–Newton method for solving $h(t) = 0$, with starting point $x_0 = -\rho$, produces the cycle

$$x_0 = -\rho, \quad x_1 = \rho, \quad x_2 = -\rho, \dots$$

As a consequence, it does not converge. Therefore, the lemma is proved. \square

Lemma 11. *The point x_* is the unique zero of F in $B(x_*, \sigma)$.*

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Proof. Assume that $y \in B(x_*, \sigma)$ and $F(y) = 0$. Using $F(x_*) = 0$ and $F(y) = 0$, we have

$$y - x_* = F'(x_*)^\dagger [F'(x_*)(y - x_*) - F(y) + F(x_*)].$$

Combining the last equation with properties of the norm and the definition of β , we obtain

$$\|y - x_*\| \leq \beta \int_0^1 \|F'(x_*) - F'(x_* + u(y - x_*))\| \|y - x_*\| du.$$

Using (2) with $x = x_* + u(y - x_*)$, $\tau = 0$ and some algebraic manipulation, we easily conclude, from the last equality, that

$$\|y - x_*\| \leq \int_0^1 [f'(u\|y - x_*\|) - f'(0)] \|y - x_*\| du = f(\|y - x_*\|) + \|y - x_*\|.$$

Since $0 < \|y - x_*\| < \sigma$, i.e., $f(\|y - x_*\|) < 0$, the last inequality implies that $\|y - x_*\| < \|y - x_*\|$, which is a contradiction. Hence, $y = x_*$. \square

2.2. Gauss-Newton sequence

In this section, we will prove the statements in Theorem 2 involving the Gauss-Newton sequence $\{x_k\}$. First, note that the first equation in (3) together with (11) implies that the sequence $\{x_k\}$ satisfies

$$x_{k+1} = G_F(x_k), \quad k = 0, 1, \dots \tag{13}$$

which is indeed an equivalent definition of this sequence.

Corollary 12. *The sequence $\{x_k\}$ is well defined, contained in $B(x_*, r)$ and it converges to the point x_* , which is the unique zero of f in $B(x_*, \sigma)$. Furthermore it holds:*

$$\lim_{k \rightarrow \infty} [\|x_{k+1} - x_*\| / \|x_k - x_*\|] = 0. \tag{14}$$

If, additionally, (h3) holds, the sequences $\{x_k\}$ and $\{t_k\}$ satisfy

$$\|x_{k+1} - x_*\| \leq [t_{k+1}/t_k^{p+1}] \|x_k - x_*\|^{p+1} \leq [t_1/t_0^{p+1}] \|x_k - x_*\|^{p+1}, \quad k = 0, 1, \dots \tag{15}$$

Consequently, for $k \geq 0$,

$$\|x_k - x_*\| \leq \begin{cases} t_0 [t_1/t_0]^k, & \text{if } p = 0; \\ t_0 [t_1/t_0]^{((p+1)^k - 1)/p}, & \text{if } p \neq 0. \end{cases}$$

Proof. Since $x_0 \in B(x_*, r) \setminus \{x_*\}$, i.e., $0 < \|x_0 - x_*\| < r$, and $r \leq \nu$, combining (13), the inclusion in Lemma 8, Lemma 6 and an induction argument, we conclude that $\{x_k\}$ is well defined and it remains in $B(x_*, r)$.

We will now prove that $\{x_k\}$ converges to x_* . Since $\|x_k - x_*\| < r \leq \rho$, for $k = 0, 1, \dots$, we obtain from (13), Lemma 8 and Proposition 4, that

$$0 \leq \|x_{k+1} - x_*\| = \|G_F(x_k) - x_*\| \leq |n_f(\|x_k - x_*\|)| \|x_k - x_*\|, \quad k = 0, 1, \dots \tag{16}$$

So, $\{\|x_k - x_*\|\}$ is a bounded and strictly decreasing sequence. Therefore $\{\|x_k - x_*\|\}$ converges. Let $\ell_* = \lim_{k \rightarrow \infty} \|x_k - x_*\|$. Since $\{\|x_k - x_*\|\}$ remains in $(0, \rho)$ and is strictly decreasing, we have $0 \leq \ell_* < \rho$. Thus, taking the limit in (16) with t converging to 0 and using the continuity of n_f in $[0, \rho)$, we obtain that $0 \leq \ell_* = |n_f(\ell_*)|$. But, if $\ell_* \neq 0$, Proposition 4 implies $|n_f(\ell_*)| < \ell_*$, hence $\ell_* = 0$. Therefore, the convergence $x_k \rightarrow x_*$ is proved. The uniqueness was proved in Lemma 11.

In order to prove the equality in (14), note that Eq. (16) implies

$$[\|x_{k+1} - x_*\| / \|x_k - x_*\|] \leq [n_f(\|x_k - x_*\|) / \|x_k - x_*\|], \quad k = 0, 1, \dots$$

Since $\lim_{k \rightarrow \infty} \|x_k - x_*\| = 0$, the desired inequality follows from the first statement in Proposition 4.

Now we will show (15). First, we will prove by induction that the sequences $\{x_k\}$ and $\{t_k\}$, defined in (13) and (7), respectively, satisfy

$$\|x_k - x_*\| \leq t_k, \quad k = 0, 1, \dots \tag{17}$$

Due to $t_0 = \|x_0 - x_*\|$, the above inequality holds for $k = 0$. Now, assume that $\|x_k - x_*\| \leq t_k$. Using (13), Lemma 9, the induction assumption and (7), we obtain that

$$\|x_{k+1} - x_*\| = \|G_F(x_k) - x_*\| \leq \frac{|n_f(t_k)|}{t_k^{p+1}} \|x_k - x_*\|^{p+1} \leq |n_f(t_k)| = t_{k+1},$$

and (17) holds. Therefore, it is easily seen that the first inequality in (15) follows by combining (13), (17), Lemma 9 and (7). The second inequality in (15) is immediate, due to the fact that the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing. Finally, for the last part of the corollary, it is enough to use (15) and some simple algebraic manipulations. \square

The proof of Theorem 2 follows from Corollary 5, the Lemmas 10 and 11 and Corollary 12.

3. Special cases

In this section, we present some special cases of Theorem 2.

3.1. Convergence results under Hölder-like and Smale conditions

In this section, we present a local convergence theorem for the Gauss–Newton method under a Hölder-like condition, see [2,9]. We also provide a Smale’s theorem on the Gauss–Newton method for analytical functions, cf. [11].

Theorem 13. Let \mathbb{X} and \mathbb{Y} be Hilbert spaces, $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \rightarrow \mathbb{Y}$ be a continuously differentiable function such that F' has a closed image in Ω . Let $x_* \in \Omega$, $R > 0$, $\beta := \|F'(x_*)^\dagger\|$ and $\kappa := \sup\{t \in [0, R) : B(x_*, t) \subset \Omega\}$. Suppose that $F(x_*) = 0$, $F'(x_*)$ is injective and there exists a constant $K > 0$ and $0 < p \leq 1$ such that

$$\beta \|F'(x) - F'(x_* + \tau(x - x_*))\| \leq K(1 - \tau^p)\|x - x_*\|^p, \quad x \in B(x_*, \kappa) \quad \tau \in [0, 1].$$

Let

$$r = \min\{\kappa, [(p + 1)/((2p + 1)K)]^{1/p}\}.$$

Then, the sequences $\{x_k\}$ and $\{t_k\}$, with starting points $x_0 \in B(x_*, r) \setminus \{x_*\}$ and $t_0 = \|x_0 - x_*\|$, respectively, such that

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad t_{k+1} = \frac{K p t_k^{p+1}}{(p + 1)[1 - K t_k^p]}, \quad k = 0, 1, \dots, \tag{18}$$

are well defined; $\{t_k\}$ is strictly decreasing, contained in $(0, r)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_*, r)$, it converges to the point x_* , which is the unique zero of F in $B(x_*, [(p + 1)/K]^{1/p})$, and there hold:

$$\|x_{k+1} - x_*\| \leq \frac{K p}{(p + 1)[1 - K t_k^p]} \|x_k - x_*\|^{p+1} \leq \frac{K p}{(p + 1)[1 - K \|x_0 - x_*\|^p]} \|x_k - x_*\|^{p+1}, \tag{19}$$

for all $k = 0, 1, \dots$, and

$$\|x_k - x_*\| \leq \left[\frac{K p \|x_0 - x_*\|^p}{(p + 1)[1 - K \|x_0 - x_*\|^p]} \right]^{[(p+1)^k - 1]/p} \|x_0 - x_*\|, \quad k = 0, 1, \dots \tag{20}$$

Moreover, if $[(p + 1)/((2p + 1)K)]^{1/p} < \kappa$, then $r = [(p + 1)/((2p + 1)K)]^{1/p}$ is the best possible convergence radius.

Proof. It is immediate to prove that F, x_* and $f : [0, \kappa) \rightarrow \mathbb{R}$, defined by $f(t) = Kt^{p+1}/(p + 1) - t$, satisfy the inequality (2) and the conditions (h1)–(h3) in Theorem 2. In this case, it is easily seen that ρ and ν , as defined in Theorem 2, satisfy

$$\rho = [(p + 1)/((2p + 1)K)]^{1/p} \leq \nu = [1/K]^{1/p},$$

and, as a consequence, $r = \min\{\kappa, [(p + 1)/((2p + 1)K)]^{1/p}\}$. Moreover, $f(\rho)/(\rho f'(\rho)) - 1 = 1$, $f(0) = f([(p + 1)/K]^{1/p}) = 0$ and $f(t) < 0$ for all $t \in (0, [(p + 1)/K]^{1/p})$. Therefore, the statements of the theorem follow from Theorem 2. \square

Remark 3. For $p = 1$ in the previous theorem, we obtain the convergence of the Gauss–Newton method for injective-overdetermined systems of equations under a Lipschitz condition, as obtained in Corollary 19 of [5].

In the following numerical example, the results of this section are illustrated.

Example 2. Let $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$. Consider the function $H : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by

$$H(x) := (ax^{4/3} - 2x, bx^{4/3} + x)^T.$$

It is easy to check that $H(0) = 0$, i.e., $x_* = 0$,

$$H'(x) = \begin{pmatrix} \frac{4}{3}ax^{1/3} - 2 \\ \frac{4}{3}bx^{1/3} + 1 \end{pmatrix}, \quad H^\dagger(x) = 9/(16(a^2 + b^2)x^{2/3} - 24(2a - b)x^{1/3} + 45) H'(x)^T,$$

and

$$\beta = \sqrt{5}/5, \quad \beta \|H'(x) - H'(\tau x)\| \leq (4\sqrt{5(a^2 + b^2)}/15) (1 - \tau^{1/3})|x|^{1/3}, \quad x \in \mathbb{R} \quad \tau \in [0, 1].$$

Hence, applying the Theorem 13 with

$$x_* = 0, \quad F = H, \quad p = 1/3, \quad K = (4\sqrt{5(a^2 + b^2)}/15), \quad r = (3/\sqrt{5(a^2 + b^2)})^3,$$

we can conclude that the sequences $\{x_k\}$ and $\{t_k\}$ as defined in (18), with starting points $x_0 \in B\left(0, \left(3/\sqrt{5(a^2 + b^2)}\right)^3\right) \setminus \{0\}$ and $t_0 = \|x_0\|$, respectively, are well defined; $\{t_k\}$ is strictly decreasing, contained in $\left(0, \left(3/\sqrt{5(a^2 + b^2)}\right)^3\right)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B\left(0, \left(3/\sqrt{5(a^2 + b^2)}\right)^3\right)$, it converges to the point x_* , which is the unique zero of F in $B\left(x_*, \left(5/\sqrt{5(a^2 + b^2)}\right)^3\right)$, and the inequalities (19) and (20) hold. Moreover, $\left(3/\sqrt{5(a^2 + b^2)}\right)^3$ is the best possible convergence radius.

Below, we present a theorem correspondent to Theorem 2 under Smale's condition, which has first appeared in Dedieu and Shub [18], see also Corollary 23 of [5].

Theorem 14. Let \mathbb{X} and \mathbb{Y} be Hilbert spaces, $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \rightarrow \mathbb{Y}$ an analytic function such that F' has a closed image in Ω . Let $x_* \in \Omega$, $R > 0$, $\beta := \|F'(x_*)^\dagger\|$ and $\kappa := \sup\{t \in [0, R) : B(x_*, t) \subset \Omega\}$. Suppose that $F(x_*) = 0$, $F'(x_*)$ is injective and

$$\gamma := \sup_{n>1} \beta \left\| \frac{F^{(n)}(x_*)}{n!} \right\|^{1/(n-1)} < +\infty.$$

Let

$$r := \min\left\{\kappa, \left(5 - \sqrt{17}\right)/(4\gamma)\right\}.$$

Then, the sequences $\{x_k\}$ and $\{t_k\}$, with starting points $x_0 \in B(x_*, r) \setminus \{x_*\}$ and $t_0 = \|x_0 - x_*\|$, respectively, such that

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad t_{k+1} = \frac{\gamma t_k^2}{2(1 - \gamma t_k)^2 - 1}, \quad k = 0, 1, \dots,$$

are well defined; $\{t_k\}$ is strictly decreasing, contained in $(0, r)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_*, r)$, it converges to the point x_* , which is the unique zero of F in $B(x_*, 1/(2\gamma))$, and there hold:

$$\|x_{k+1} - x_*\| \leq \frac{\gamma}{2(1 - \gamma t_k)^2 - 1} \|x_k - x_*\|^2 \leq \frac{\gamma}{2(1 - \gamma \|x_0 - x_*\|)^2 - 1} \|x_k - x_*\|^2, \quad k = 0, 1, \dots,$$

and

$$\|x_k - x_*\| \leq \left[\frac{\gamma \|x_0 - x_*\|}{2(1 - \gamma \|x_0 - x_*\|)^2 - 1} \right]^{(2^k - 1)} \|x_0 - x_*\|, \quad k = 0, 1, \dots$$

Moreover, if $(5 - \sqrt{17})/(4\gamma) < \kappa$, then $r = (5 - \sqrt{17})/(4\gamma)$ is the best possible convergence radius.

Proof. In this case, the real function, $f : [0, 1/\gamma) \rightarrow \mathbb{R}$, defined by $f(t) = t/(1 - \gamma t) - 2t$, is a majorant function for the function F on $B(x_*, 1/\gamma)$. Hence, as f has a convex derivative, the proof follows the same pattern as outlined in Theorem 20 of [5]. \square

3.2. Convergence result under a generalized Lipschitz condition

In this section, we present a local convergence theorem for the Gauss–Newton method under a generalized Lipschitz condition according to X. Wang (see [9,12]). It is worth to point out that the result in this section does not assume that the function defining the generalized Lipschitz condition is nondecreasing.

Theorem 15. Let \mathbb{X} and \mathbb{Y} be Hilbert spaces, $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \rightarrow \mathbb{Y}$ be a continuously differentiable function such that F' has a closed image in Ω . Let $x_* \in \Omega$, $R > 0$, $\beta := \|F'(x_*)^\dagger\|$ and $\kappa := \sup\{t \in [0, R) : B(x_*, t) \subset \Omega\}$. Suppose that $F(x_*) = 0$, $F'(x_*)$ is injective and there exists a positive integrable function $L : [0, R) \rightarrow \mathbb{R}$ such that

$$\beta \|F'(x) - F'(x_* + \tau(x - x_*))\| \leq \int_{\tau\|x-x_*\|}^{\|x-x_*\|} L(u)du, \tag{21}$$

for all $\tau \in [0, 1]$, $x \in B(x_*, \kappa)$. Let positive constants

$$\begin{aligned} \bar{\nu} &:= \sup\left\{t \in [0, R) : \int_0^t L(u)du - 1 < 0\right\}, \\ \bar{\rho} &:= \sup\left\{t \in (0, \delta) : \int_0^t L(u)u du / \left[t\left(1 - \int_0^t L(u)du\right)\right] < 1, t \in (0, \delta)\right\}, \quad \bar{r} = \min\{\kappa, \bar{\rho}\}. \end{aligned}$$

Then, the sequences $\{x_k\}$ and $\{t_k\}$, with starting point $x_0 \in B(x_*, \bar{r})/\{x_*\}$ and $t_0 = \|x_0 - x_*\|$, respectively, such that

$$x_{k+1} = x_k - F'(x_k)^{\dagger}F(x_k), \quad t_{k+1} = \int_0^{t_k} L(u)u \, du / \left(1 - \int_0^{t_k} L(u) \, du\right), \quad k = 0, 1, \dots,$$

are well defined, $\{t_k\}$ is strictly decreasing, contained in $(0, \bar{r})$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_*, \bar{r})$, it converges to x_* , which is the unique zero of F in $B(x_*, \bar{\sigma})$, where

$$\bar{\sigma} := \sup \left\{ t \in (0, \kappa) : \int_0^t L(u)(t-u) \, du - t < 0 \right\},$$

and there hold: $\lim_{k \rightarrow \infty} t_{k+1}/t_k = 0$ and $\lim_{k \rightarrow \infty} [\|x_{k+1} - x_*\|/\|x_k - x_*\|] = 0$. Moreover, if

$$\int_0^{\bar{\rho}} L(u)u \, du / \left[\bar{\rho} \left(1 - \int_0^{\bar{\rho}} L(u) \, du\right) \right] = 1,$$

and $\bar{\rho} < \kappa$, then $\bar{r} = \bar{\rho}$ is the best possible convergence radius.

If, additionally, given $0 \leq p \leq 1$

(h) the function $(0, \nu) \ni t \mapsto t^{1-p}L(t)$ is nondecreasing,

then the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing and we have:

$$\|x_{k+1} - x_*\| \leq [t_{k+1}/t_k^{p+1}] \|x_k - x_*\|^{p+1} \leq [t_1/t_0^{p+1}] \|x_k - x_*\|^{p+1}, \quad k = 0, 1, \dots$$

Consequently, for $k \geq 0$,

$$\|x_k - x_*\| \leq \begin{cases} t_0[t_1/t_0]^k, & \text{if } p = 0; \\ t_0[t_1/t_0]^{((p+1)k-1)/p}, & \text{if } p \neq 0. \end{cases}$$

Proof. Let $\bar{f} : [0, \kappa) \rightarrow \mathbb{R}$ be a differentiable function defined by

$$\bar{f}(t) = \int_0^t L(u)(t-u) \, du - t. \tag{22}$$

Note that, the derivative of the function f is given by

$$\bar{f}'(t) = \int_0^t L(u) \, du - 1.$$

Since L is integrable, \bar{f}' is continuous (in fact \bar{f}' is absolutely continuous). Hence, it is easy to see that (21) becomes (2) with $f' = \bar{f}'$. Moreover, since L is positive, the function $f = \bar{f}$ satisfies the conditions (h1) and (h2) in Theorem 2. Direct algebraic manipulation yields

$$\frac{1}{t^{p+1}} \left[\frac{\bar{f}(t)}{\bar{f}'(t)} - t \right] = \left[\frac{1}{t^{p+1}} \int_0^t L(u)u \, du \right] \frac{1}{|\bar{f}'(t)|}.$$

If assumption (h) holds, then Lemma 2.2 of [13] implies that the first term on the right hand side of the above equation is nondecreasing in $(0, \nu)$. Now, since $1/|\bar{f}'|$ is strictly increasing in $(0, \nu)$, the above equation implies that (h3) in Theorem 2, with $f = \bar{f}$, also holds. Therefore, the result follows from Theorem 2 with $f = \bar{f}$, $\nu = \bar{\nu}$, $\rho = \bar{\rho}$, $r = \bar{r}$ and $\sigma = \bar{\sigma}$. □

Remark 4. If the positive integrable function $L : [0, R) \rightarrow \mathbb{R}$ is nondecreasing, then the strictly increasing function $f' : [0, R) \rightarrow \mathbb{R}$, defined by

$$f'(t) = \int_0^t L(u) \, du - 1,$$

is convex. Hence, the sequence generated by the Gauss–Newton method converges with quadratic rate, see for example Corollary 8 of [5]. Moreover, in this case it is not hard to prove that the inequalities (2) and (21) are equivalent. However, if f' is strictly increasing and not necessarily convex, the inequalities (2) and (21) are not equivalent, because there exist continuous and strictly increasing functions with derivative zero almost everywhere. These functions are not absolutely continuous, i.e., they cannot be represented by an integral, see examples in [19,20].

4. Final remarks

The inexact Gauss–Newton like methods for solving (1) are described as follows: Given an initial point $x_0 \in \Omega$, define

$$x_{k+1} = x_k + S_k, \quad B(x_k)S_k = -F'(x_k)^*F(x_k) + r_k, \quad k = 0, 1, \dots,$$

where $B(x_k)$ is a suitable invertible approximation of the derivative $F'(x_k)^*F'(x_k)$, the residual tolerance, r_k , and the preconditioning invertible matrix, P_k , are such that

$$\|P_k r_k\| \leq \theta_k \|P_k F'(x_k)^*F(x_k)\|,$$

for a suitable forcing number θ_k . It would be interesting to study this class of methods under a majorant condition, without the convexity assumption on the derivative of the majorant function. This analysis will be carried out in the future.

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