Retratos de Fase Globais e Integrabilidade de Sistemas Holomorfos

Paulo Ricardo da Silva
Universidade Estadual Paulista

Colaboradores: L.F.S. Gouveia Gabriel Rondon

Esquema da apresentação:

1- Sistemas Holomofos (overview)

2- Integrabilidade Local

3- Retratos de Fase Globais

4- Sistemas Holomorfos por partes

1- Sistemas Holomorfos

$$\dot{z} = f(z), \quad z \in \mathbb{C}, \quad t \in \mathbb{R}$$

onde $f(z)$ é holomorfa numa vizinhança perfurada D de $z_0 \in \mathbb{C}$.

Escrevendo como sistema planar:

$$\dot{x} = u(x, y)$$
$$\dot{y} = v(x, y)$$

Onde u, v satisfazem as equações de Cauchy-Riemann equations em D:

$$u_x = v_y, \quad u_y = -v_x.$$

Propriedades dos Sistemas Holomorfos: $\dot{z} = f(z) = u(x,y) + iv(x,y)$

1) Se $f \neq 0$ então os equilíbrios são isolados.

2) Sistemas holomorfos não possuem ciclos limites.

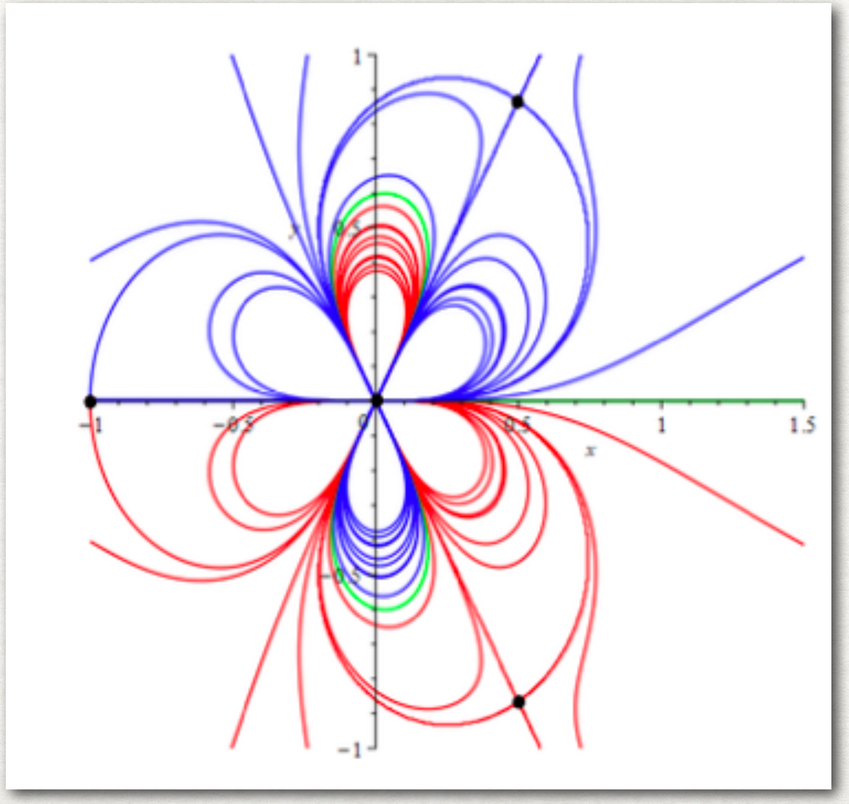
3) $f : D \to \mathbb{C}, \qquad f(z) = \sum_{i=1}^{\infty} \frac{B_k}{z^k} + \sum_{i=1}^{\infty} A_k z^k$

$A_k = a_k + ib_k, \quad B_k = c_k + id_k, \quad (x+iy)^k = p_k + iq_k$

$$\begin{cases} \dot{x} &= \sum_{k=1}^{\infty} \left(c_k \frac{p_k}{(x^2+y^2)^k} + d_k \frac{q_k}{(x^2+y^2)^k} \right) + a_0 + \sum_{k=1}^{\infty} (a_k p_k - b_k q_k) \\ \dot{y} &= \sum_{k=1}^{\infty} \left(d_k \frac{p_k}{(x^2+y^2)^k} - c_k \frac{q_k}{(x^2+y^2)^k} \right) + b_0 + \sum_{k=1}^{\infty} (b_k p_k + a_k q_k) \end{cases}$$

Em particular sistemas holomorfos polinomiais de grau n depende somente de $2n+2$ parametros.

4) Se $f(z) \in \mathbb{R}, \forall z \in \mathbb{R}$ então o retrato de fase é simétrico com respeito ao eixo x. Para ver isso basta notar que nesse caso $d_k = b_k = 0$ e portanto em x' teremos somente polinômios do tipo p_k que são pares em y. Isso segue também diretamente do Princípio de Reflexão de Schwarz



Retratos Locais em torno de z_0 (equilíbrio ou singularidade)

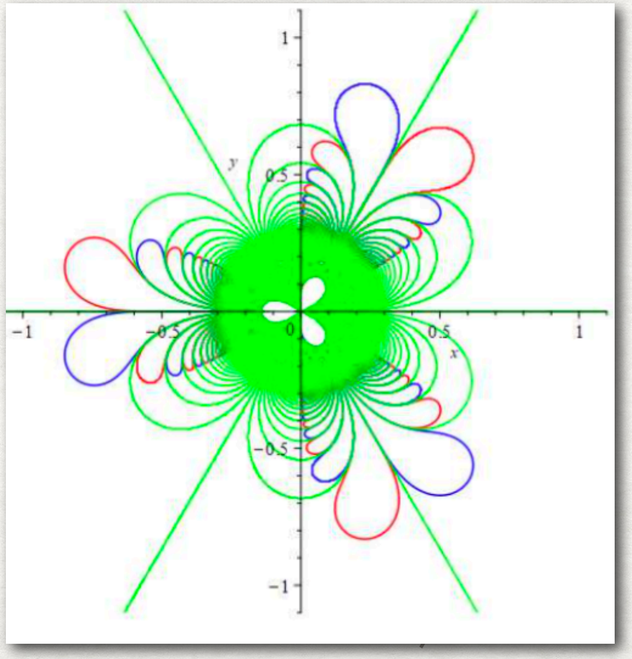
A conjugação usada para classificar o retrato de fase local é a conjugação conforme (uma aplicação conforme que leva o retrato de fase de $f(z)$ em torno de z_0 no retrato de fase de $g(z)$ em torno de 0.)

(a) Ponto regular:	$f(z_0) \neq 0$	$g(z) = 1$
(b) Zero simples	$f(z) = a_1(z - z_0) + \ldots, \quad a_1 \neq 0$	$g(z) = a_1 z$
(c) Zero de ordem n :	$f(z) = a_n(z - z_0)^n + \ldots, \quad a_n \neq 0$ $\mathrm{Res}(1/f, z_0) = 0$	$g(z) = z^n$
	$f(z) = a_n(z - z_0)^n + \ldots, \quad a_n \neq 0$ $\mathrm{Res}(1/f, z_0) = 1/\gamma$	$g(z) = \dfrac{\gamma z^n}{1 + z^{n-1}}$
(d) Polo de ordem n :	$f(z) = \dfrac{a_n}{(z - z_0)^n} + \ldots + a_0 + \ldots$	$g(z) = \dfrac{1}{(z - z_0)^n}$

Teorema. Seja $\dot{z} = f(z)$ um sistema holomorfo definido na vizinhança de uma singularidade essencial z_0. Dado qualquer direção w_0, existe z, arbitrariamente próximo de z_0, cuja trajetória começando em z segue a direção w, com w arbitrariamente próximo de w_0.

$$|z - z_0| < \varepsilon, \quad |w - w_0| < \delta, \quad \frac{d}{dt}\varphi_f(0,z) = f(z) = w\,.$$

Example. $\dot{z} = z^4 e^{\frac{1}{z^3}}$



2- Integrabilidade Local de Sistemas Holomorfos

$$\dot{z} = f(z) = u(x,y) + iv(x,y)$$

O conjugado de um sistema holomorfo é um sistema gradiente.

$$\dot{z} = \overline{f(z)} = u(x,y) - iv(x,y) = \nabla\phi = (\phi_x, \phi_y)$$

onde $\phi = \Re(F)$ sendo $F(z)$ uma primitiva de $f(z)$. De fato

$$F(z) = \phi(x,y) + i\psi(x,y)$$

$$F'(z) = f(z) \implies F'(z) = \phi_x - i\phi_y = u + iv$$

Teorema. $\dot{z} = \overline{f(z)}$ é gradiente e suas trajetórias estão contidas nas curvas de nível de ψ onde $F = \phi + i\psi$ é uma primitiva de f.

Como $\dot{z} = \overline{f(z)}$ é o gradiente de ϕ, suas trajetórias são ortogonais as curvas de nível de ϕ. Além disso $F = \phi + i\psi$ ié holomorfa e então as curvas de nível de ψ e de ϕ são ortogonais.

Corolário As órbitas de $\dot{z} = f(z)$ estão contidas nas curvas de nível de $\Im G(z)$

Com $G'(z) = \dfrac{1}{f(z)}$.

O retrato de fase de $\dot{z} = f(z) = \dfrac{|f(z)|}{\overline{f(z)}}$ é o mesmo de $\dot{z} = \dfrac{\overline{1}}{f(z)}$ onde $f(z) \neq 0$.

Uma maneira formal de obter a integral primeira:

$$\frac{dz}{dt} = f(z) \implies \int \frac{dz}{f(z)} = \int dt = (t + a) + bi$$

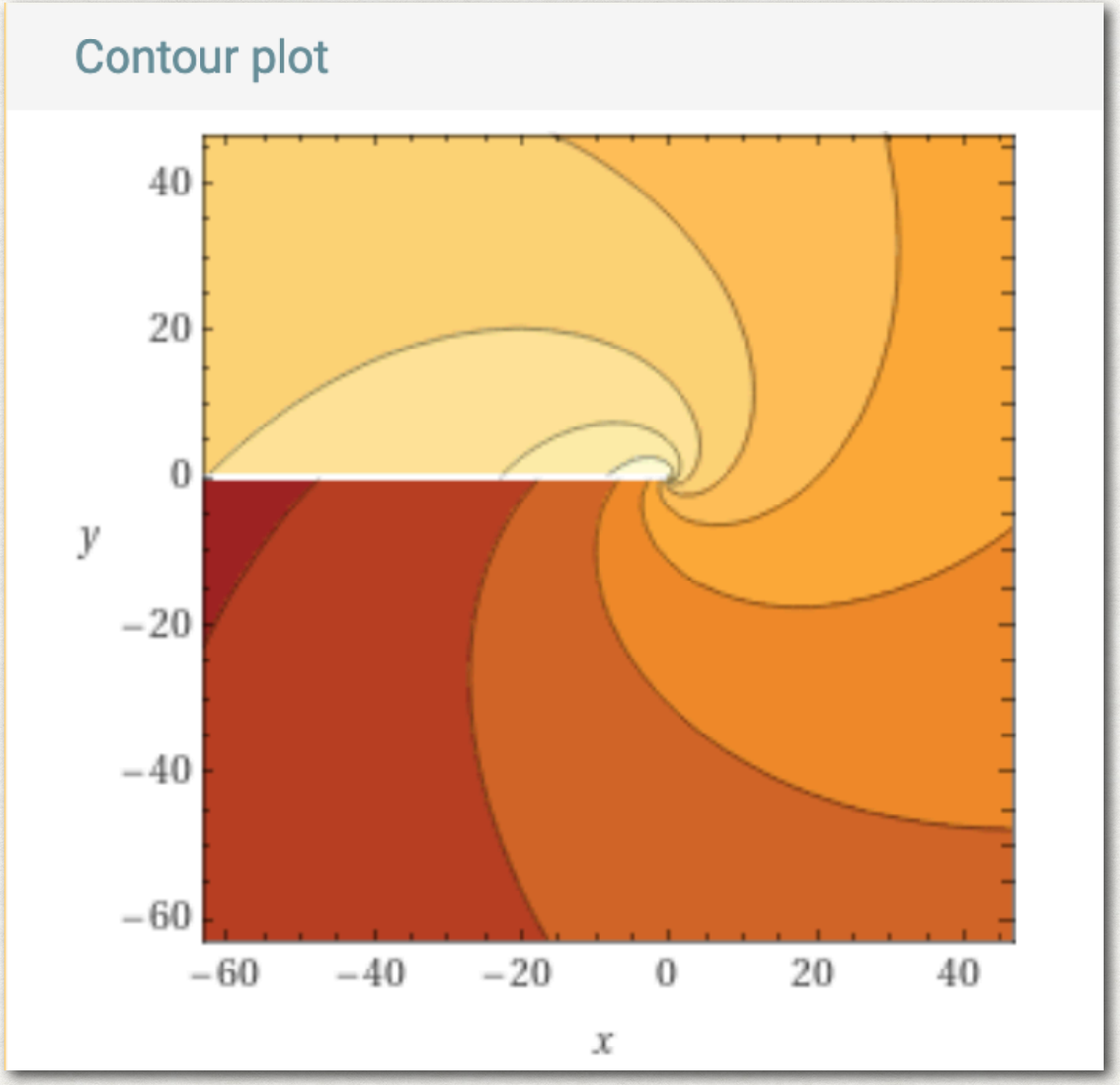
$$\Im\left(\int \frac{dz}{f(z)}\right) = b\,.$$

Exemplo.

$$\dot{z} = (1+i)z$$

$$\int \frac{dz}{(1+i)z} = \left(\frac{1}{2} - \frac{i}{2}\right) \log z + const\,.$$

$$H(x,y) = \arctan\frac{y}{x} - \log\sqrt{x^2+y^2}$$



Formas normais e integrais primeiras

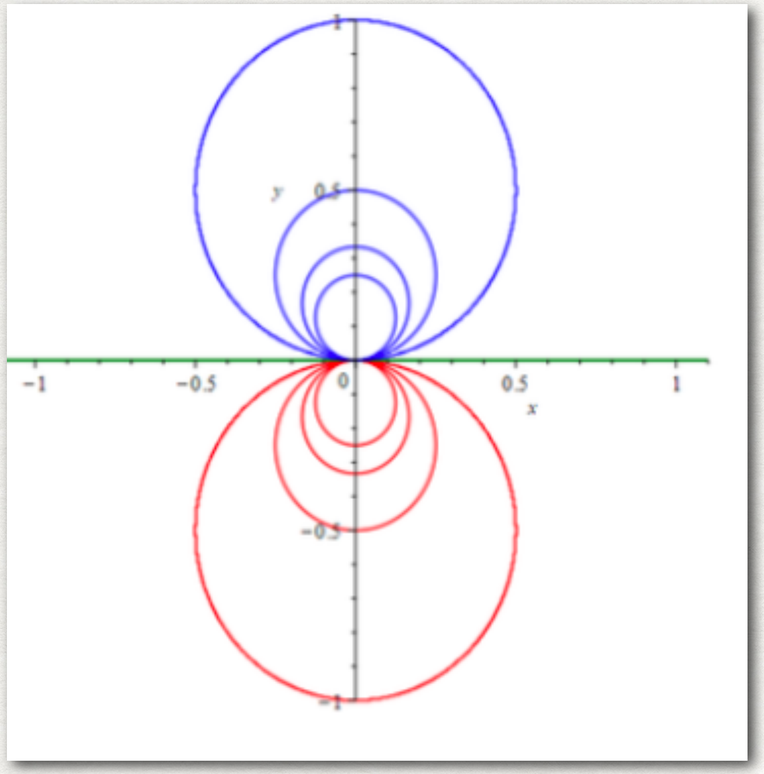
$$\dot{z} = 1$$

$$H(x, y) = y$$

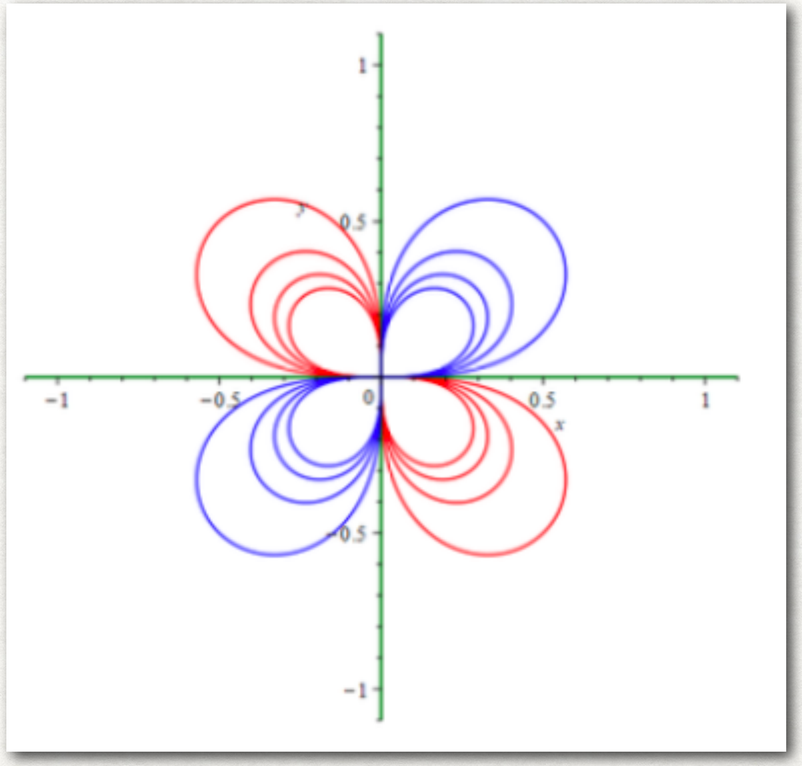
$$\dot{z} = z^s$$

$$\int \frac{dz}{z^s} = \frac{z^{1-s}}{1-s} + const\,.$$

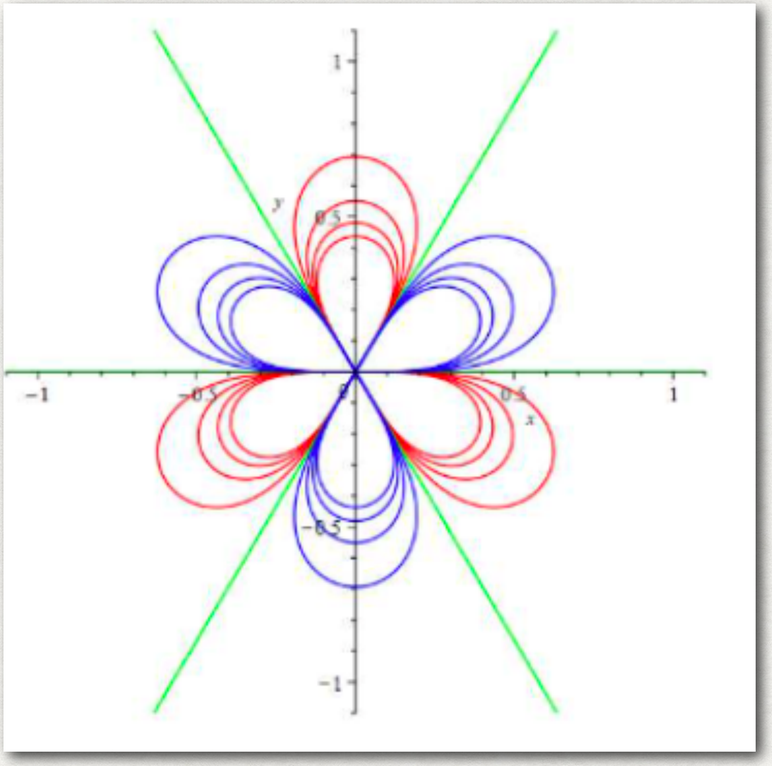
$$H(x, y) = (x^2 + y^2)^{\frac{1-s}{2}} \sin((1-s)\arctan\frac{y}{x})$$



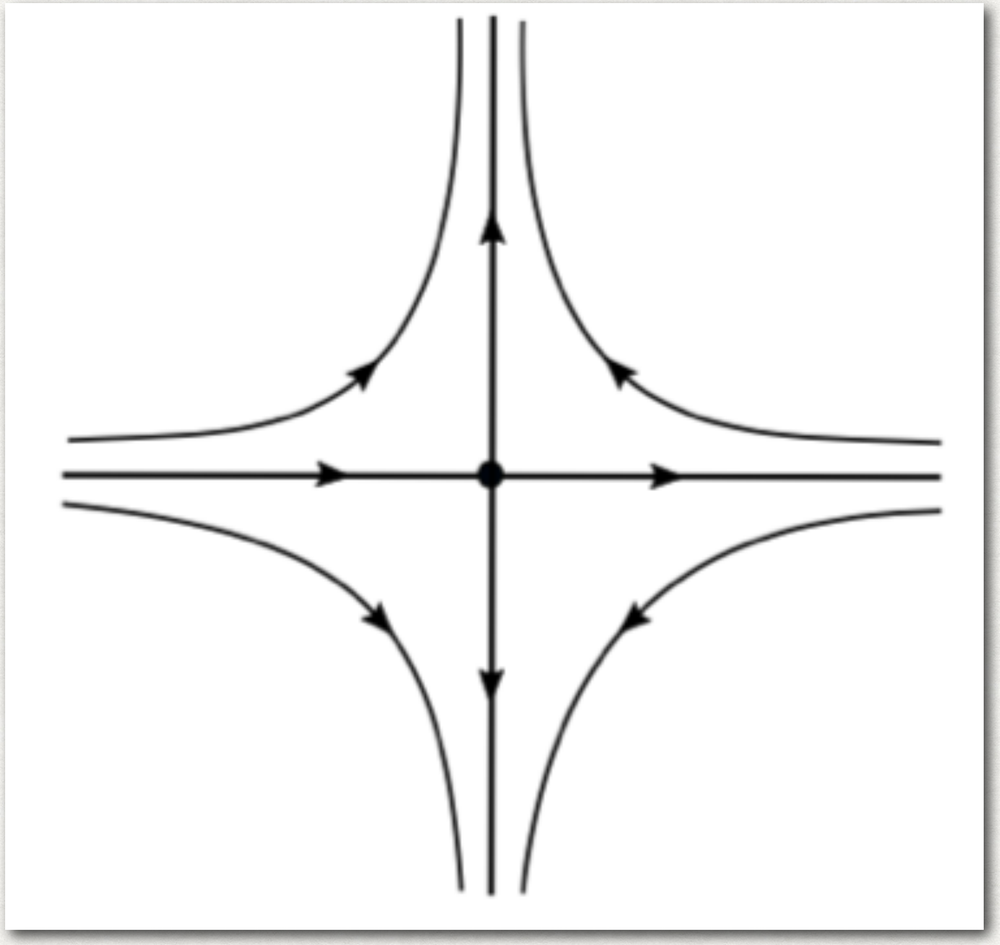
$\dot{z} = z^2$



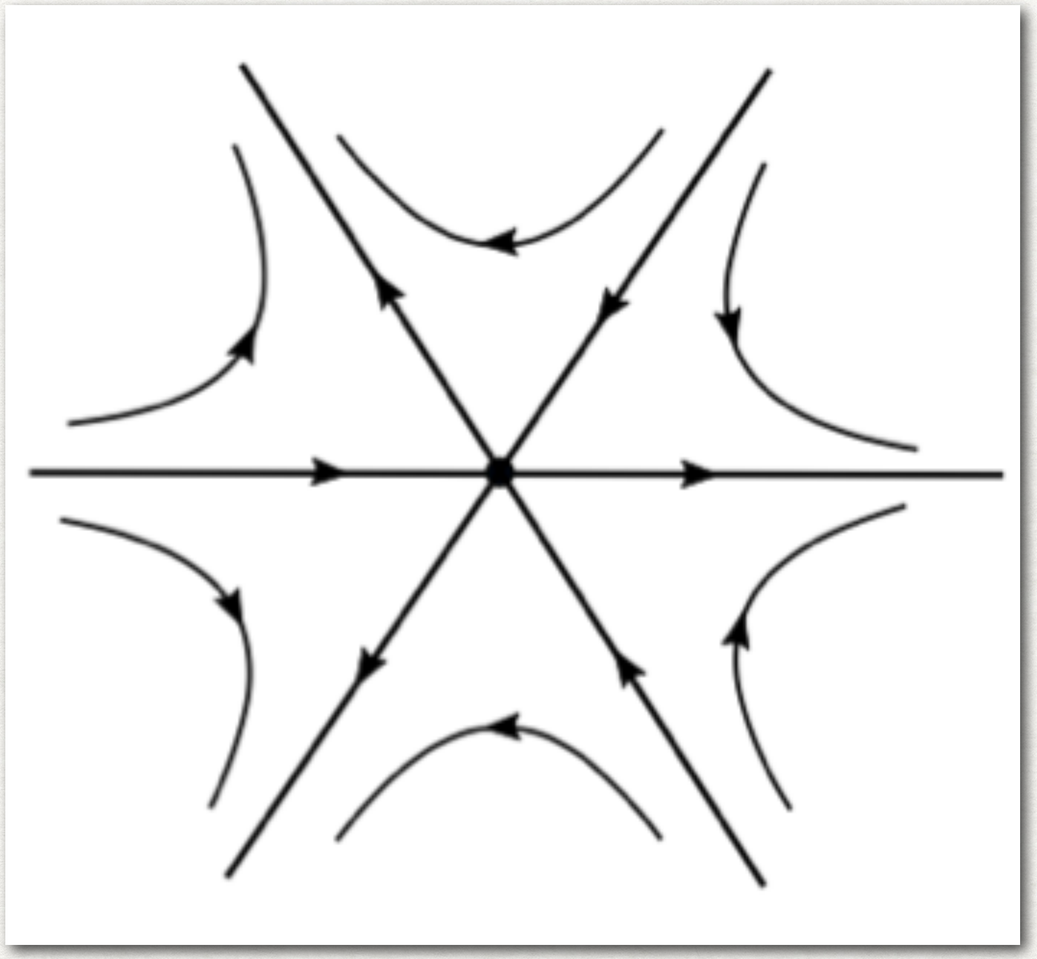
$\dot{z} = z^3$



$\dot{z} = z^4$



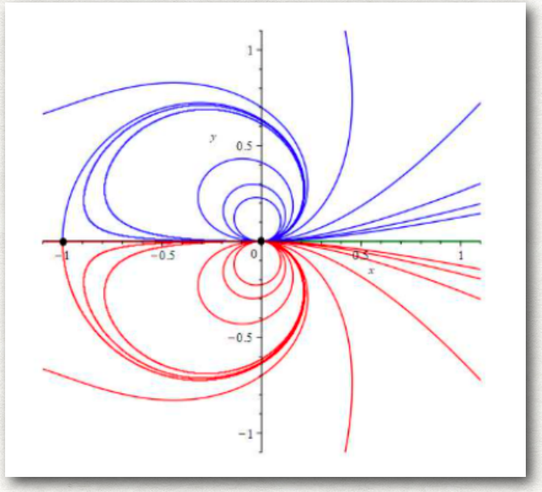
$$\dot{z} = \frac{1}{z}$$



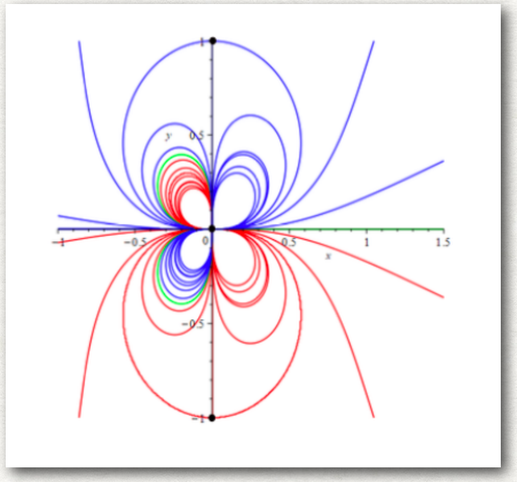
$$\dot{z} = \frac{1}{z^2}$$

$$\dot{z} = \frac{(a+bi)z^n}{1+z^{n-1}}$$

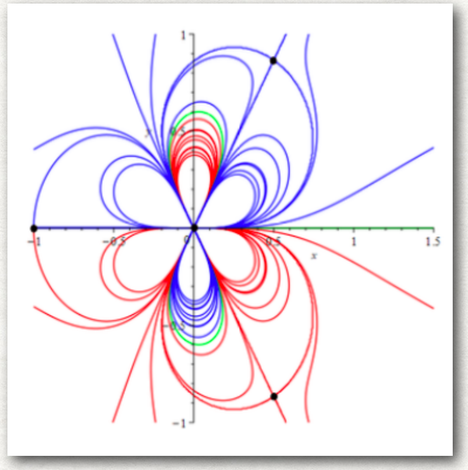
$$H(x,y) = \frac{a}{1-n}(x^2+y^2)^{\frac{1-n}{2}}\sin((1-s)\arctan\frac{y}{x}) - \frac{b}{1-n}(x^2+y^2)^{\frac{1-n}{2}}\cos((1-s)\arctan\frac{y}{x}) +$$
$$+a\arctan\frac{y}{x} - b\log\sqrt{x^2+y^2}$$



$$\dot{z} = \frac{z^2}{1+z}$$



$$\dot{z} = \frac{z^3}{1+z^2}$$

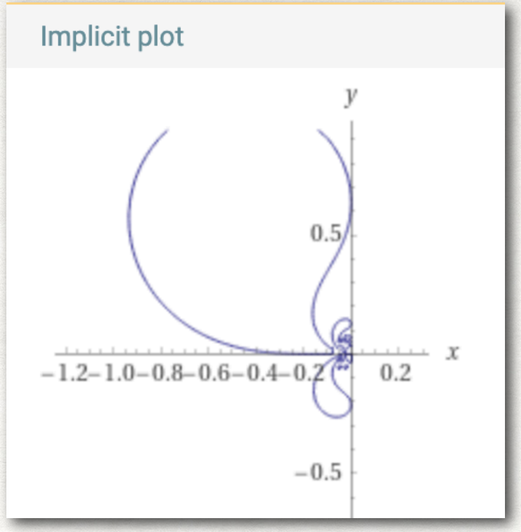


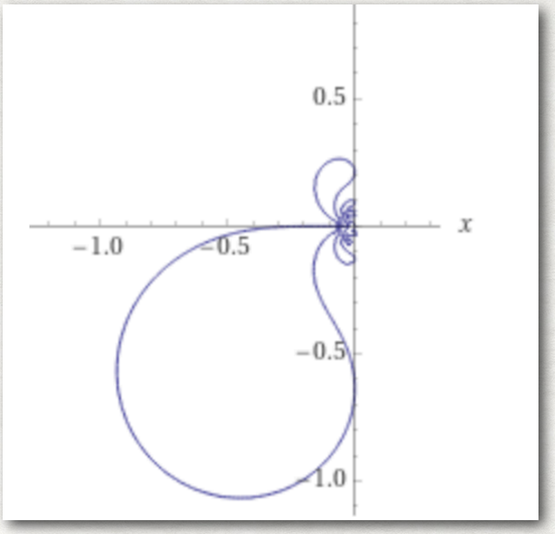
$$\dot{z} = \frac{z^4}{1+z^3}$$

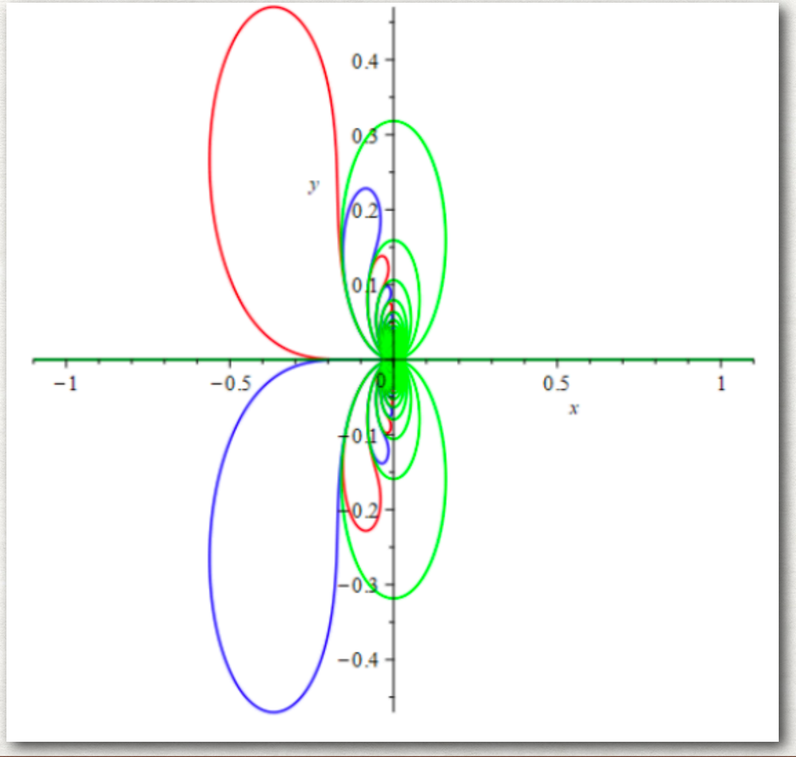
Exemplo. $\dot{z} = z^2 e^{\frac{1}{z}}$

$$\int \frac{dz}{z^2 e^{\frac{1}{z}}} = e^{\frac{-1}{z}} + const\,.$$

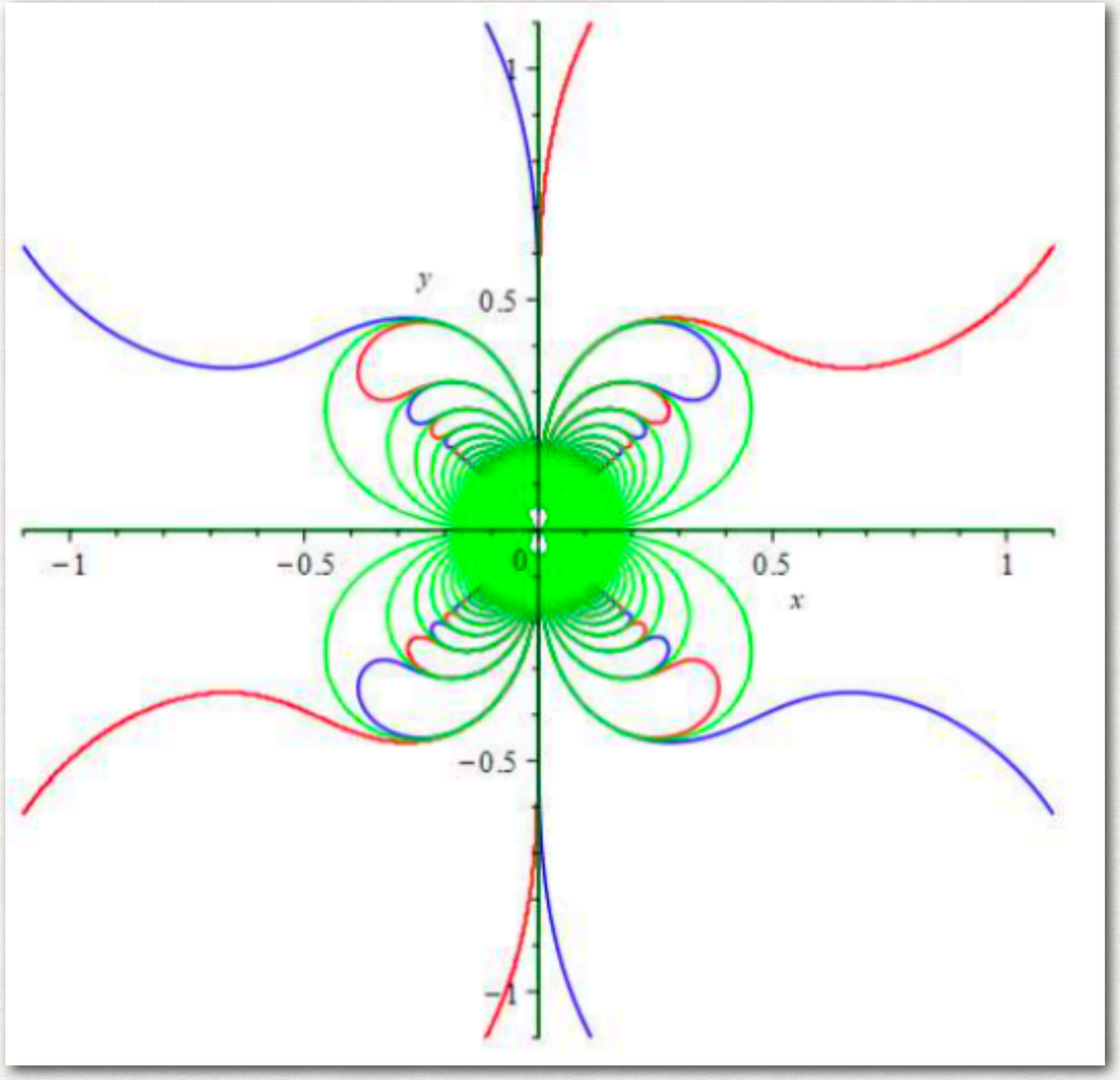
$$H(x,y) = e^{\frac{-x}{x^2+y^2}} \sin\left(\frac{y}{x^2+y^2}\right)$$







Exemplo. $\dot{z} = z^3 e^{\frac{1}{z^2}}$



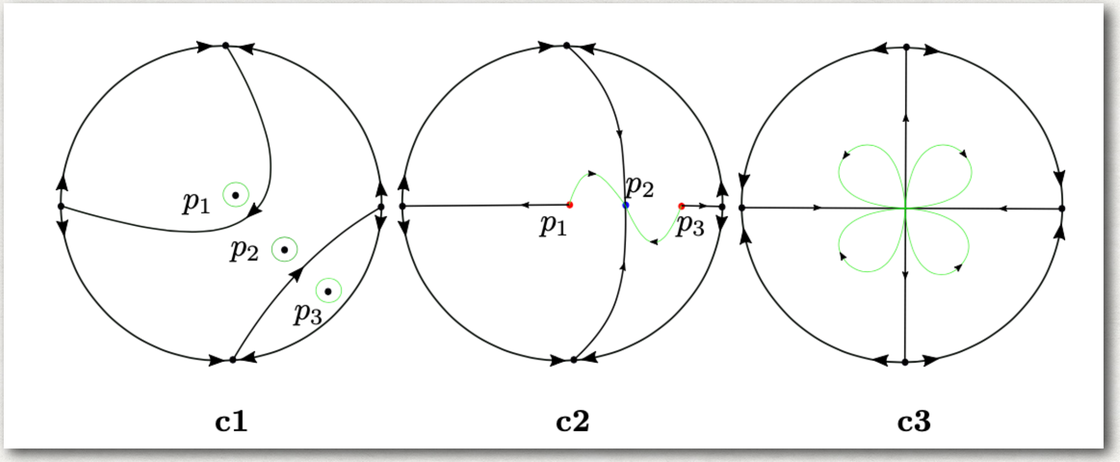
3- Retratos de fase Globais

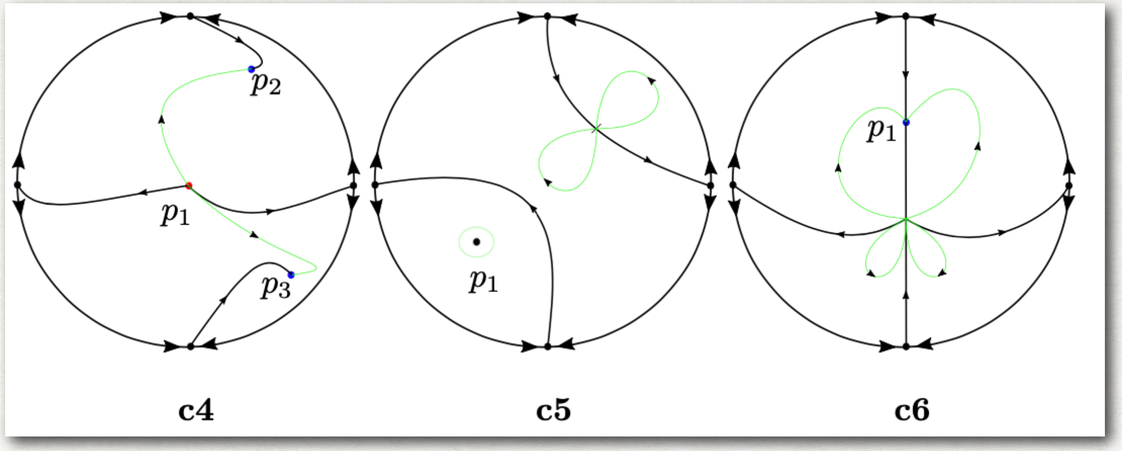
$$\dot{z} = f(z), \qquad \dot{z} = \frac{1}{f(z)} \quad \text{and} \quad \dot{z} = T(z)$$

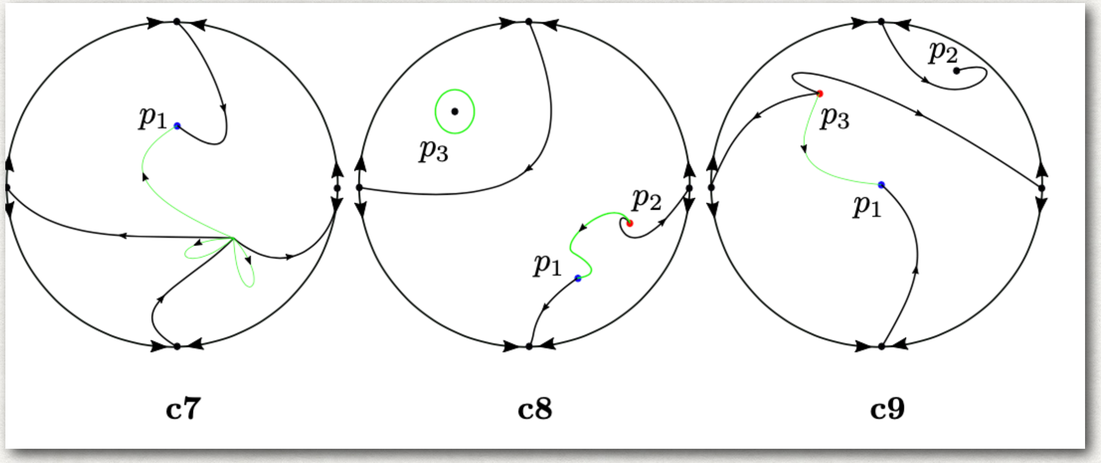
where f is a complex polynomial of degree $2, 3, 4$ and $T(z) = (Az + B)/(Cz + D)$ is a Moebius transformation, that is, $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$. Moreover, we provide the following possible global phase portraits on the Poincaré disk, unless topological equivalence:

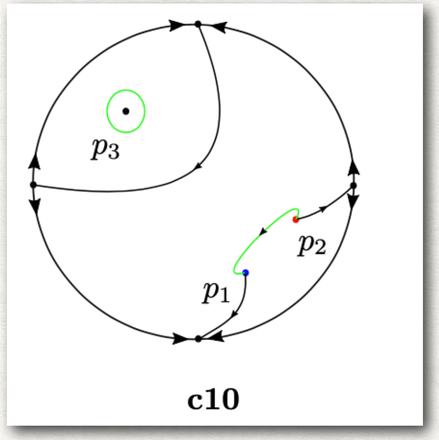
- If f is a complex polynomial of degree 3 (resp 4) then there are 10 (resp 30) possible phase portraits of $\dot{z} = f(z)$.

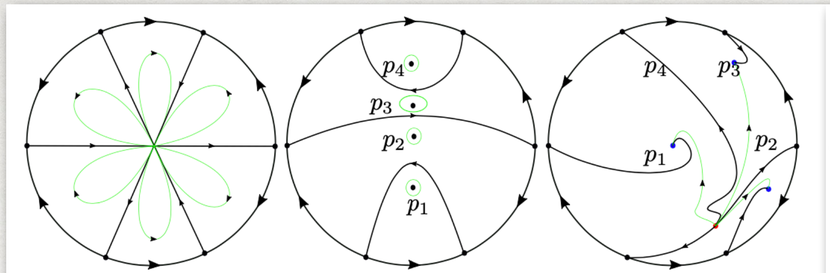
- If f is a complex polynomial of degree 3 (resp 4) then there are 4, (resp 11) possible phase portraits of $\dot{z} = \frac{1}{f(z)}$.
- If T is a Moebius transformation then there are 10 possible phase portraits of $\dot{z} = T(z)$.


p_1
p_2
p_3
c1
p_1
p_2
p_3
c2
c3


p_2
p_1
p_3
c4
p_1
c5
p_1
c6

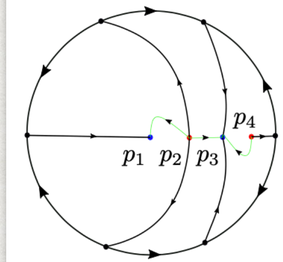

p_1
c7
p_3
p_2
p_1
c8
p_2
p_3
p_1
c9

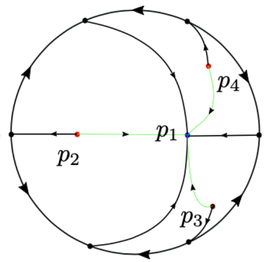

p_3
p_2
p_1
c10

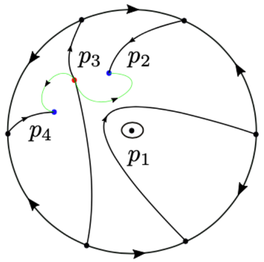

Q1

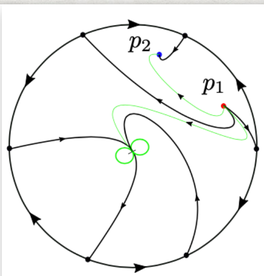
p_4
p_3
p_2
p_1
Q2

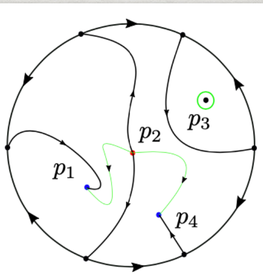
p_4
p_3
p_1
p_2
Q3

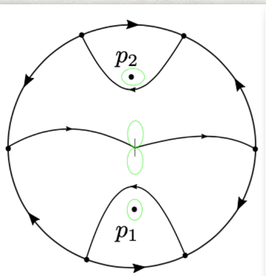

p_1
p_2
p_3
p_4
Q4

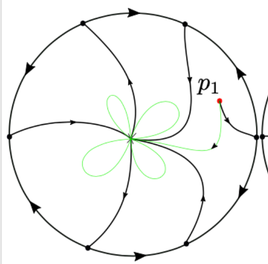

p_2
p_1
p_4
p_3
Q5)

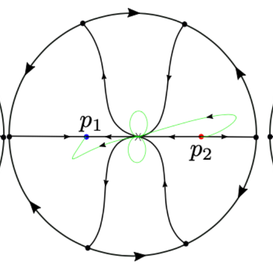

p_3
p_2
p_4
p_1
Q6

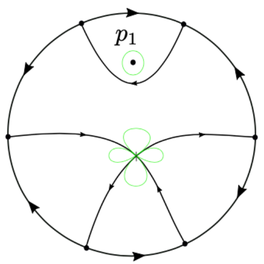

p_2
p_1
Q7

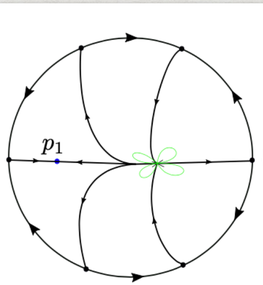

p_2
p_3
p_1
p_4
Q8

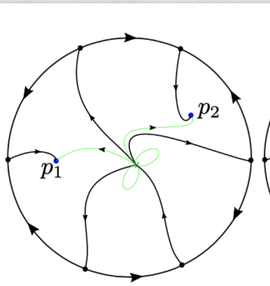

p_2
p_1
Q9)

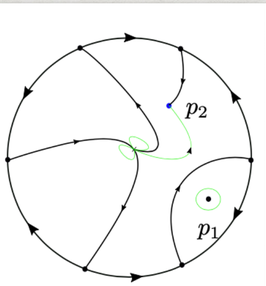

p_1
Q10

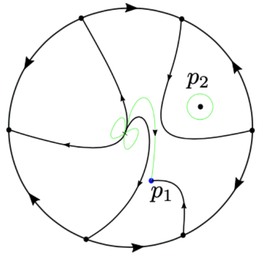

p_1
p_2
Q11

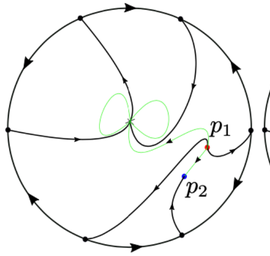

p_1
Q12

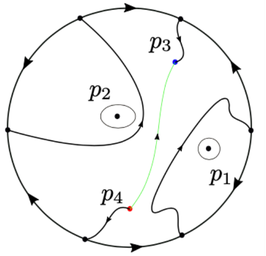

p_1
Q13

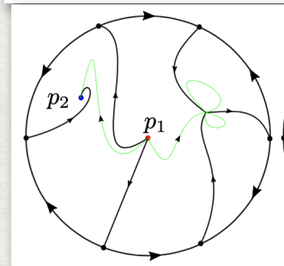

p_2
p_1
Q14

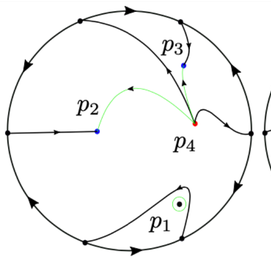

p_2
p_1
Q15

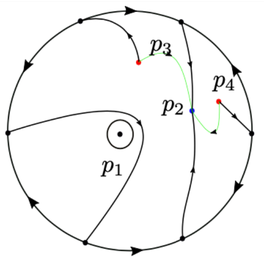

p_2
p_1
Q16

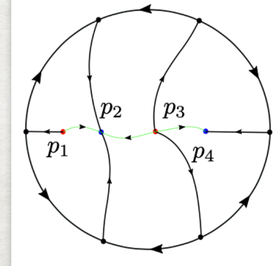

p_1
p_2
Q17

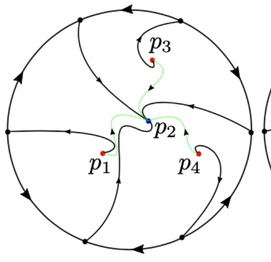

p_3
p_2
p_1
p_4
Q18

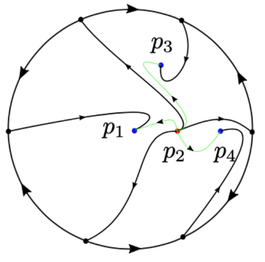

p_2
p_1
Q19

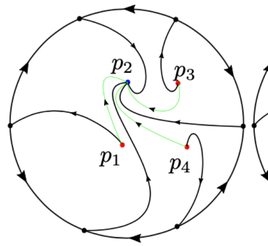

p_3
p_2
p_4
p_1
Q20

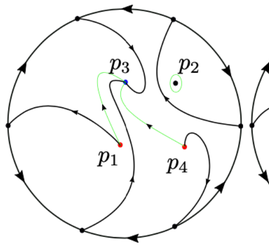

p_3
p_4
p_2
p_1
Q21

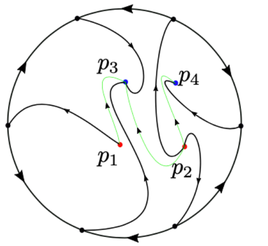

p_2
p_3
p_1
p_4
Q22

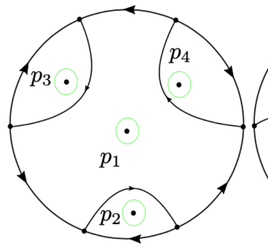

p_3
p_2
p_1
p_4
Q23

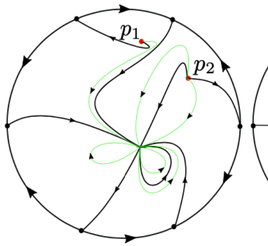

p_3
p_1
p_2
p_4
Q24

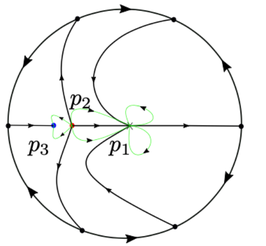

p_2
p_3
p_1
p_4
Q25

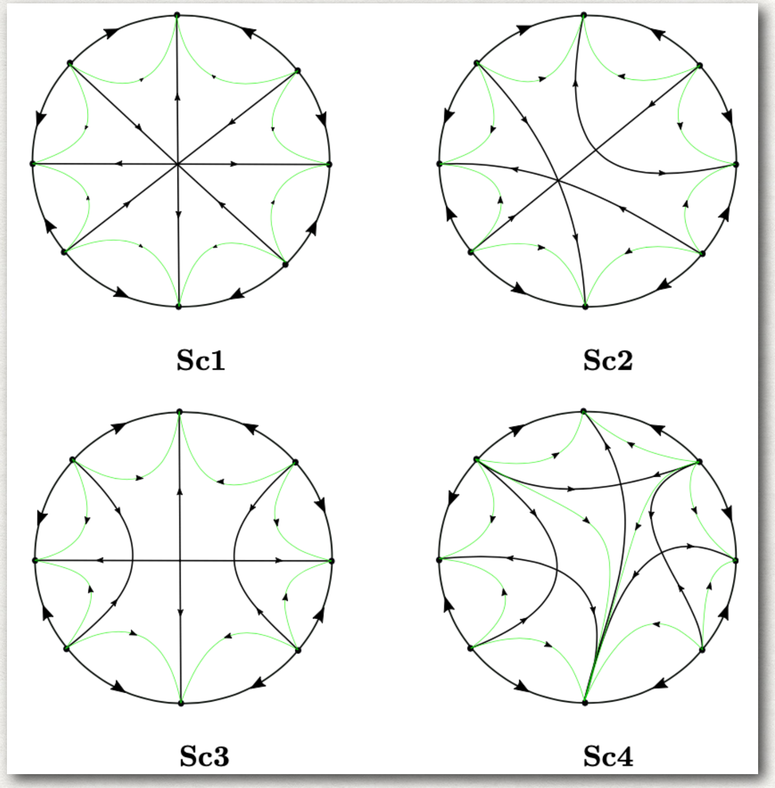

p_3
p_2
p_1
p_4
Q26

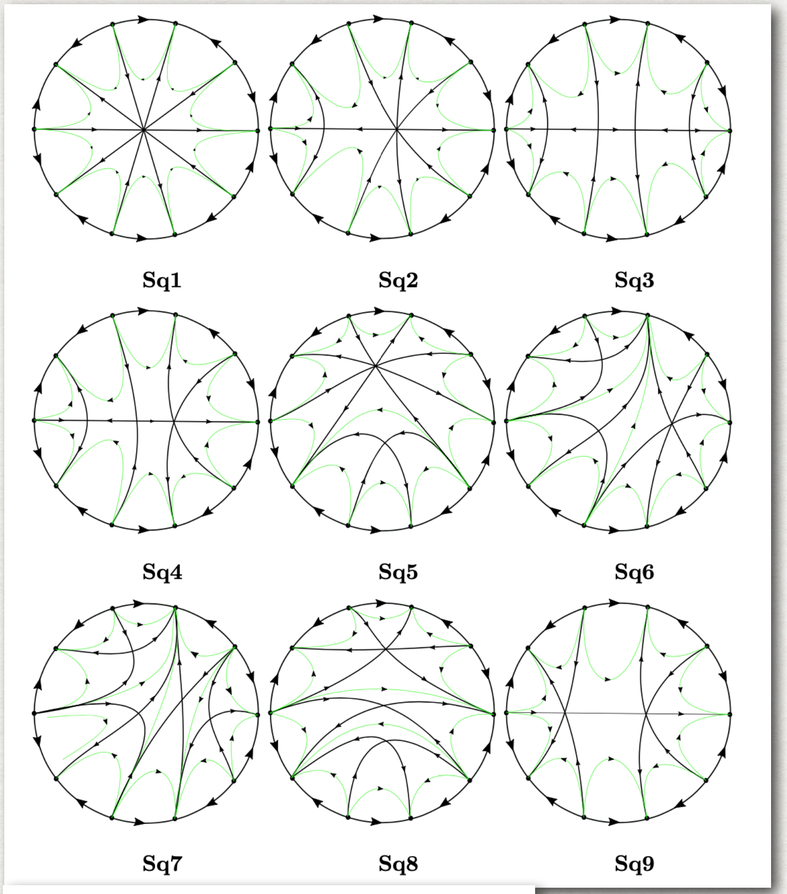

p_3
p_4
p_1
p_2
Q27

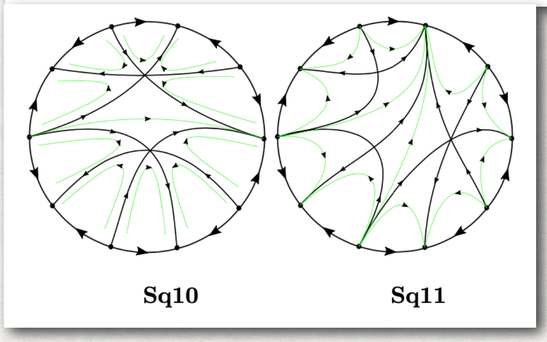

p_3
p_4
p_1
p_2
Q28

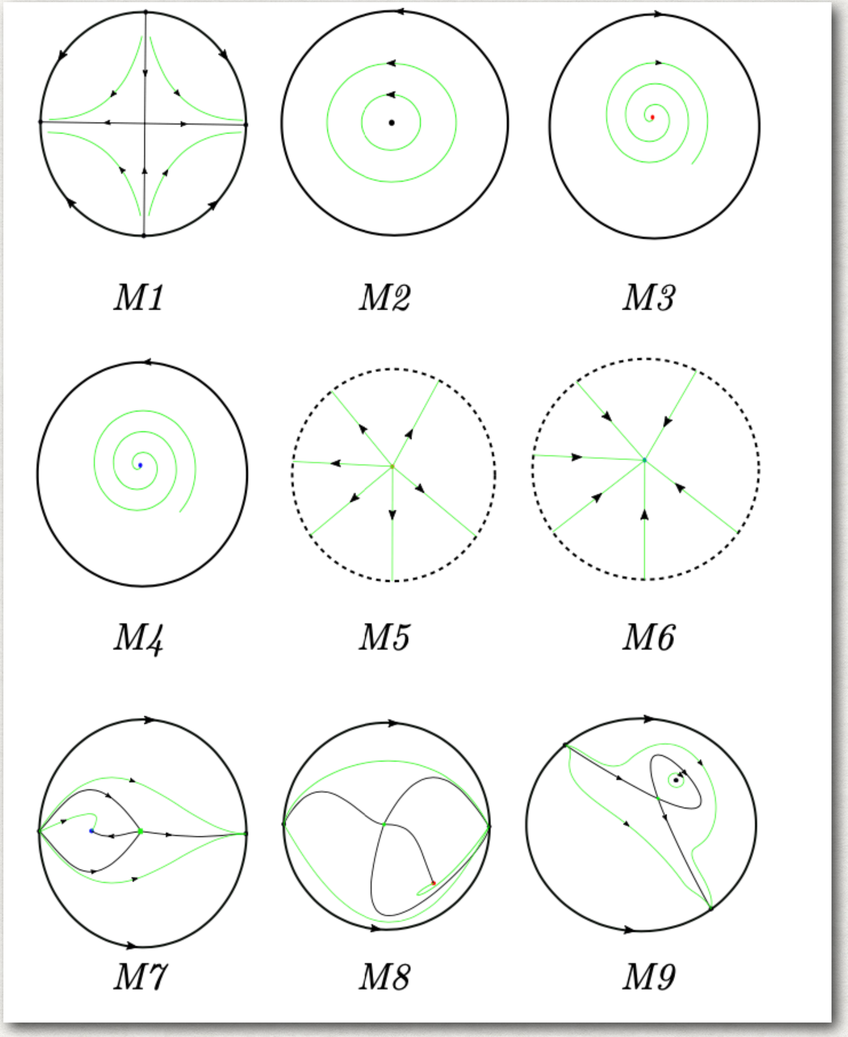

p_1
p_2
Q29


p_2
p_3
p_1
Q30


Sc1
Sc2
Sc3
Sc4


Sq1
Sq2
Sq3
Sq4
Sq5
Sq6
Sq7
Sq8
Sq9


Sq10
Sq11


M1
M2
M3
M4
M5
M6
M7
M8
M9

4- SISTEMAS HOLOMORFOS POR PARTES

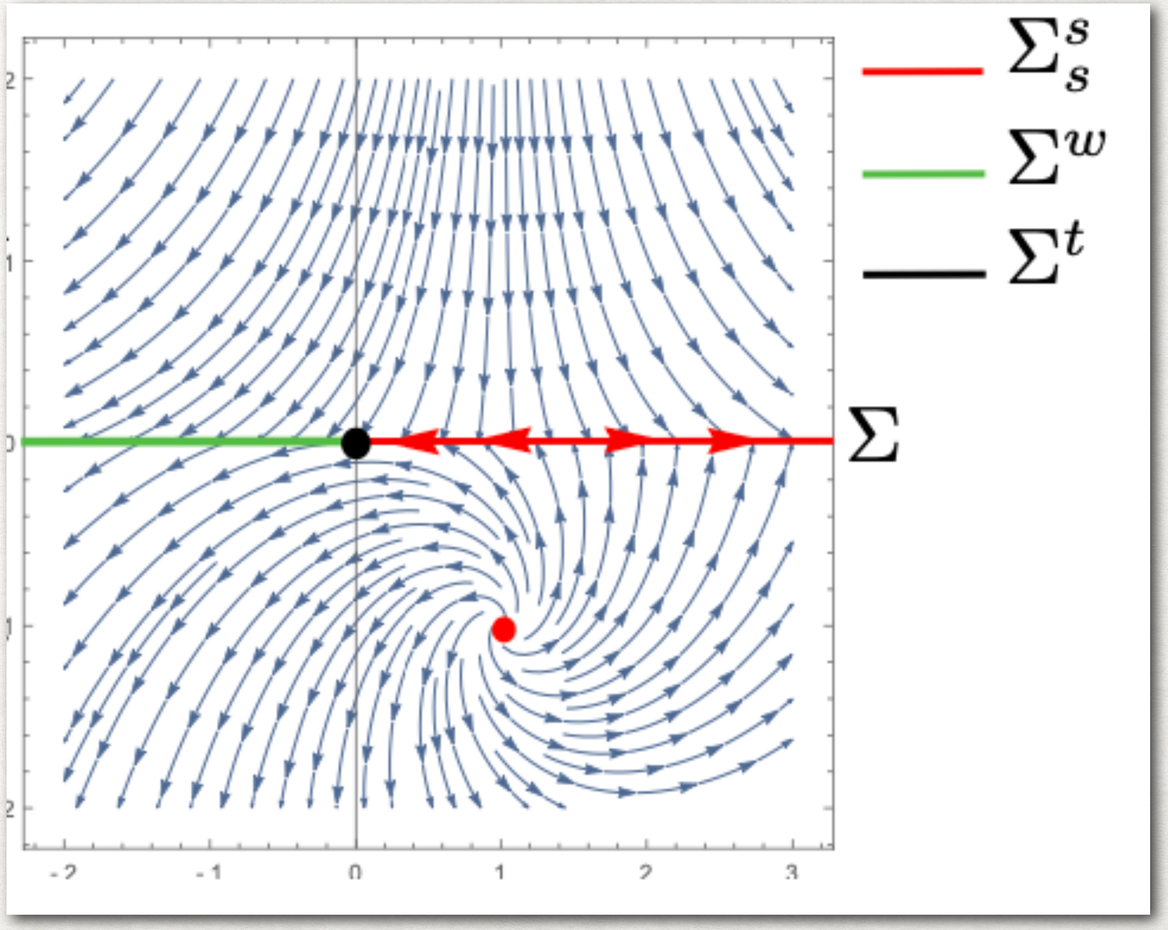
$$\dot{z} = \left(\frac{f+g}{2}\right) + \operatorname{sgn}(y)\left(\frac{f-g}{2}\right)$$

$$f = u_1 + iv_1, \quad g = u_2 + iv_2, \quad PSVF : (f, g)$$

$$\Sigma = \{y = 0\}$$

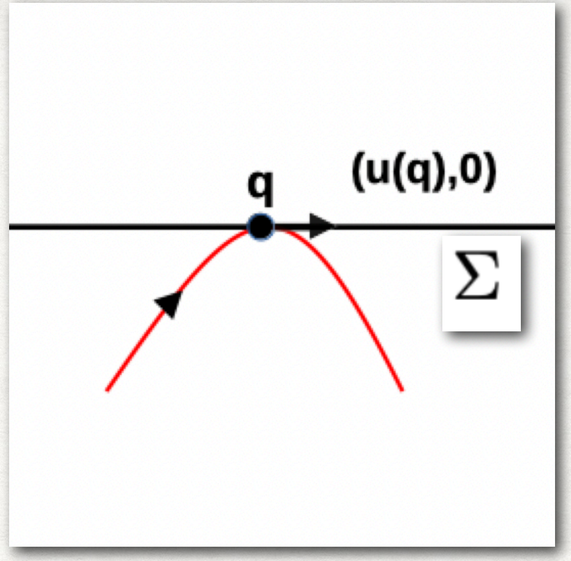
$$\Sigma^+ = \{y > 0\} \qquad \Sigma^- = \{y < 0\}$$

$$\Sigma^c = \{v_1 v_2 > 0\}$$
$$\Sigma^s = \{v_1 v_2 < 0\}$$
$$\Sigma^t = \{v_1 v_2 = 0\}$$



PONTOS DE CONTATO $\Sigma^t = \{v_1 v_2 = 0\}$

$f = u + iv, \quad f' = u_x + iv_x, \quad u(q) \neq 0, \quad v(q) = 0$



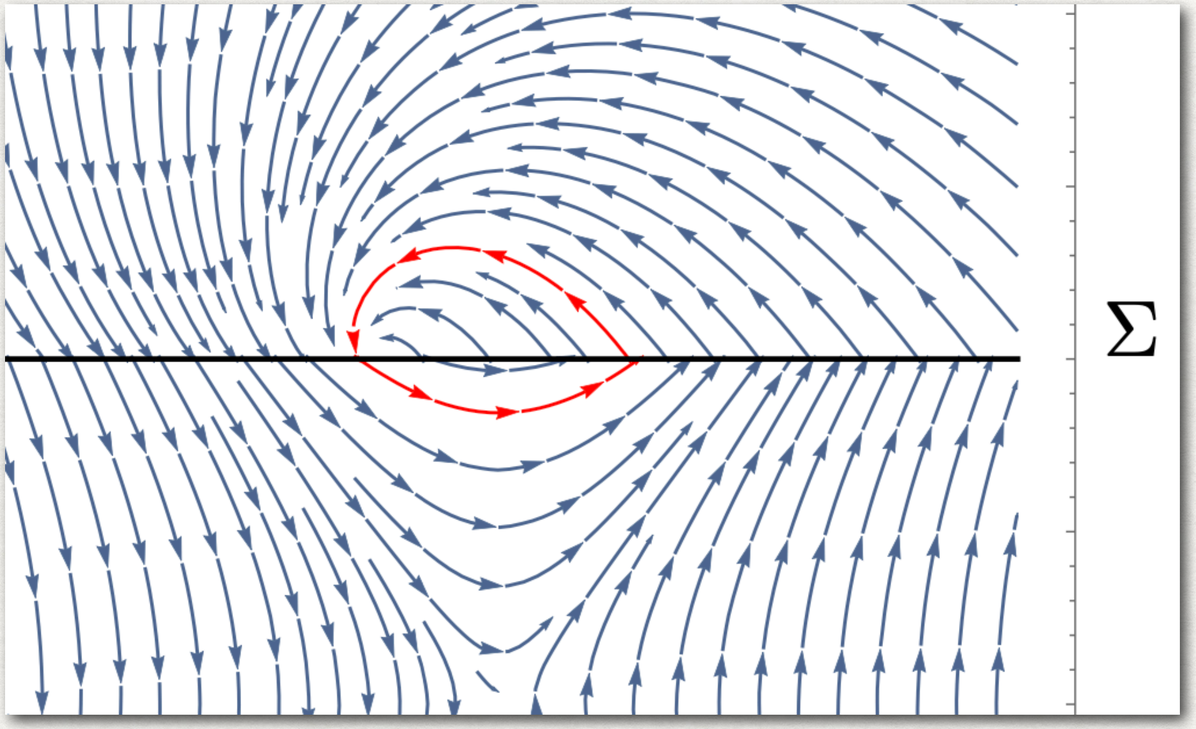
$q \in \Sigma^t$ dobra $v_x(q) \neq 0$

$q \in \Sigma^t$ contato de ordem k $\dfrac{\partial^i}{\partial y^i} v_x(q) = 0, \quad i = 0,\ldots,k-3, \quad \dfrac{\partial^{k-2}}{\partial y^{k-2}} v_x(q) \neq 0$

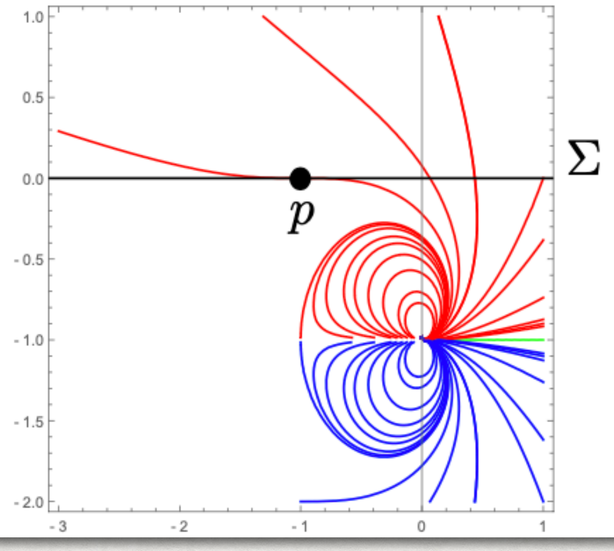
Conjugação conforme preserva pontos de deslize, de costura e de contato.

Pontos de contato das formas normais

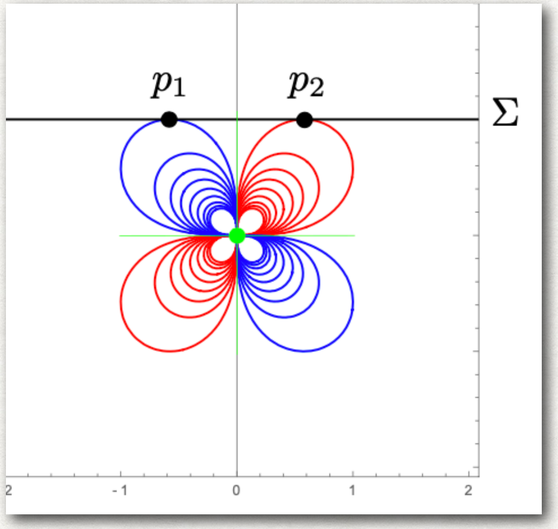
$G(z) = (a + bi)(z - z_0), \quad b \neq 0$: somente dobras

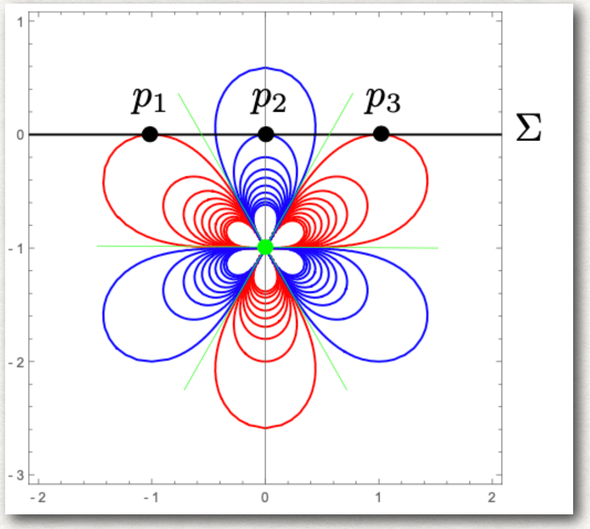


$$\dot{z} = \frac{(z+i)^2}{1+(z+i)},$$

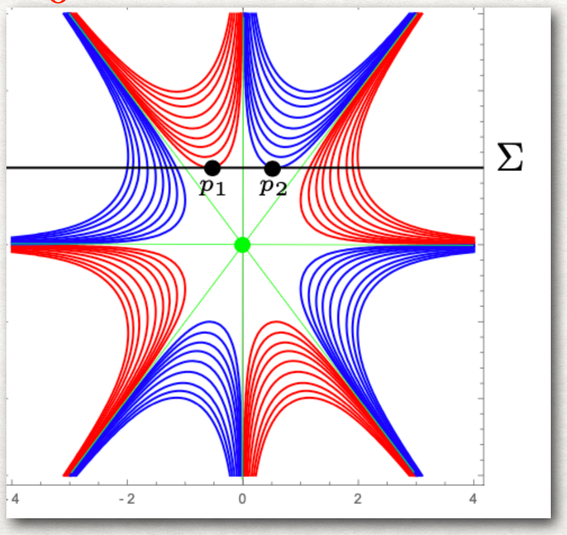


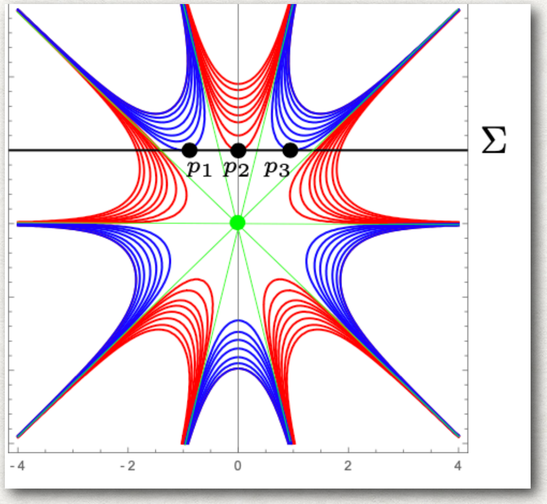
$G(z) = (z - z_0)^n$: somente dobras





$$G(z) = \frac{1}{(z - z_0)^n}$$: somente dobras





Regularização de Sistemas Holomorfos por partes

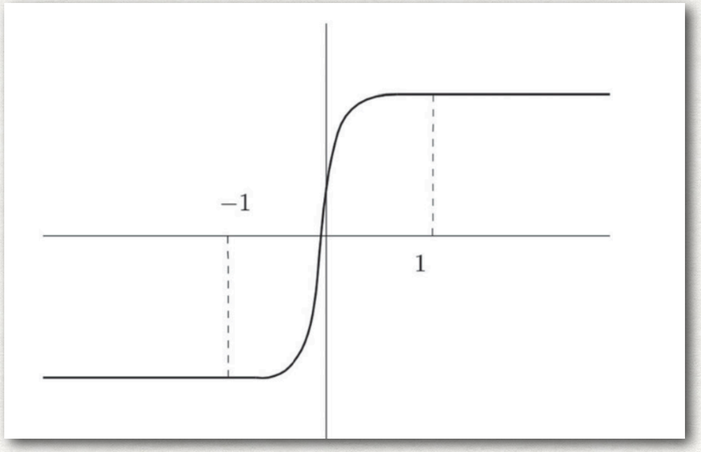
$$\dot{z} = f^+(z), \quad \dot{z} = f^-(z), \quad z = x + iy$$

$$\dot{z} = F(z) = \left(\frac{1+\lambda}{2}\right) f^+(z) + \left(\frac{1-\lambda}{2}\right) f^-(z)$$

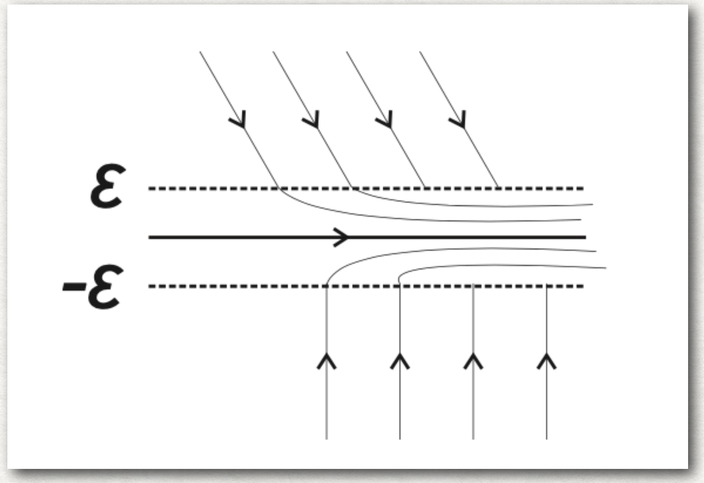
$$\lambda = \operatorname{sgn} y$$

$$\dot{z} = F^\varepsilon(z) = \left(\frac{1+\lambda}{2}\right) f^+(z) + \left(\frac{1-\lambda}{2}\right) f^-(z)$$

$$\lambda = \varphi\left(\frac{y}{\varepsilon}\right)$$



Função de transição φ



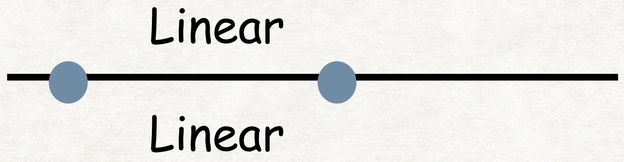
Teorema.

Se existe $\varepsilon > 0$ tal que F^ε é holomorfa então $f^+ = f^-$.

Ciclos Limites de Sistemas Holomorfos por partes

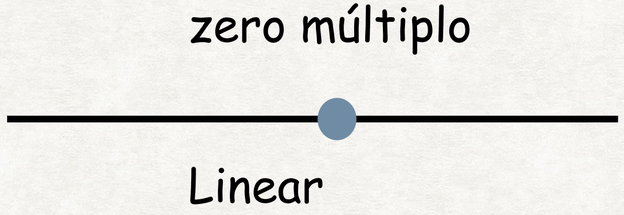
Nós consideramos pares de sistemas holomorfos dados em suas formas normais.

$$\dot{z}^+ = (a+bi)(z-x_0) \quad y>0$$
$$\dot{z}^- = (c+di)z \quad y<0$$



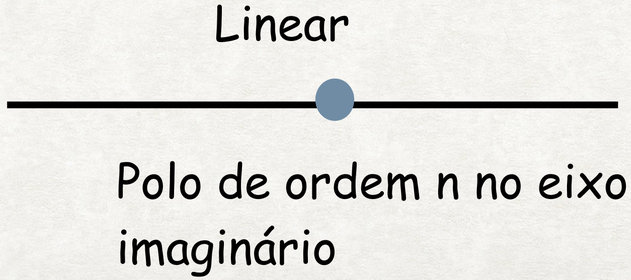
$$\dot{z}^+ = i^m(z+z_0)^n \quad y>0$$
$$\dot{z}^- = (a+bi)(z-d) \quad y<0$$

n par: m=0
n ímpar: m=1

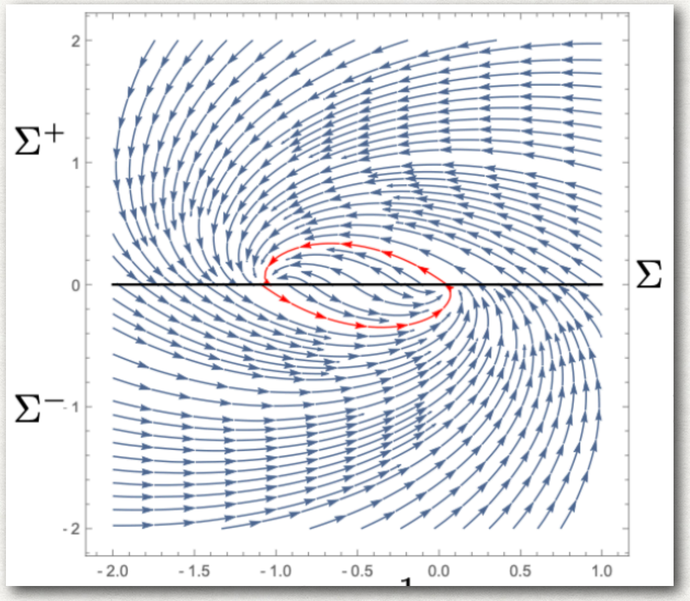


$$\dot{z}^+ = (a+bi)(z-d) \quad y>0$$
$$\dot{z}^- = \frac{i^m}{(z+iy_0)^n} \quad y<0$$

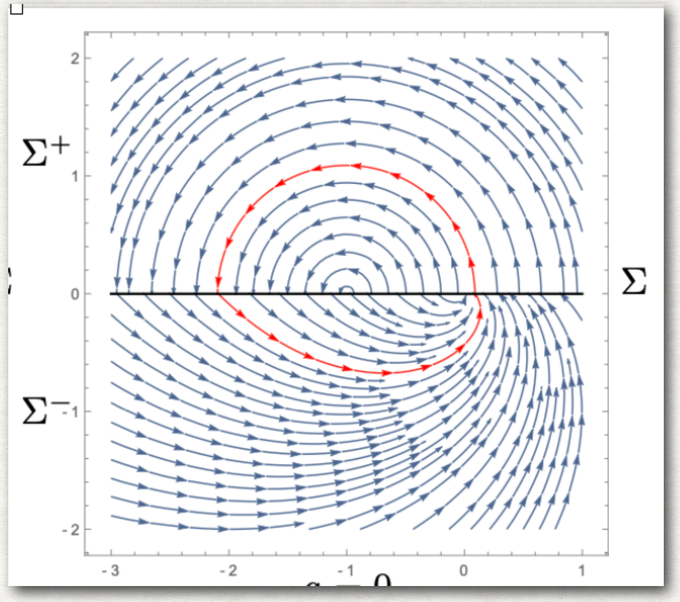
n par: m=0
n ímpar: m=1



Teorem. Os sistemas holomorfos lineares por partes cujos equilíbrios Pertencem a Σ tem no máximo um ciclo limite



$$\dot{z}^+ = (-1+i)(z+1) \quad y > 0$$
$$\dot{z}^- = (-1+i)z \qquad y < 0$$

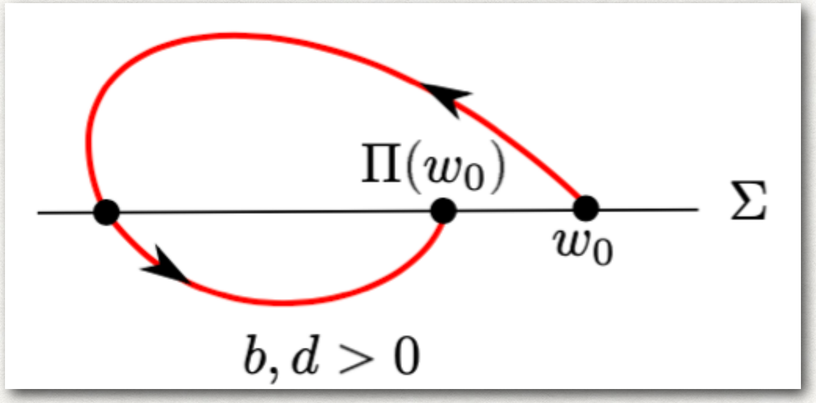


$$\dot{z}^+ = i(z+1) \quad y > 0$$
$$\dot{z}^- = (-1+i)z \qquad y < 0$$

Prova.

Assumimos $b\,.\,d \neq 0$.

$$\dot{z}^+ = (a+bi)(z-x_0) \quad y>0$$
$$\dot{z}^- = (c+di)z \qquad\quad y<0$$



Para fixar as ideias consideremos b>0. Neste caso temos

$$z^+(t) = (w_0 - x_0)e^{(a+bi)t} + x_0, \quad z^+(0) = w_0, \quad z^+(\pi/b) = -(w_0 - x_0)e^{a\pi/b} + x_0$$

$$z^-(t) = -e^{ct}((w_0 - x_0)e^{a\pi/b} - x_0)e^{idt}, \quad z^-(0) = -(w_0 - x_0)e^{a\pi/b} + x_0$$

$$z^-(\frac{\pi}{d}) = e^{\frac{c\pi}{d}}(w_0 - x_0)e^{a\pi/b} - x_0$$

$$\pi(z) = e^{c\pi/d}((z-x_0)e^{a\pi/b} - x_0) \quad \pi'(w_0) = e^{(a/b+c/d)\pi} \quad \pi(w) = w \implies w = \frac{e^{\frac{c\pi}{d}}(1+e^{\frac{a\pi}{b}})x_0}{-1+e^{a/b+c/d}\pi}$$

a	b	c	d	x_0
$+$	$+$	$+$	$+$	$+$
$-$	$+$	$-$	$+$	$-$
$-$	$-$	$-$	$-$	$+$
$+$	$-$	$+$	$-$	$-$
$+$	$+$	$-$	$+$	$\mathrm{sgn}(x_0) = \mathrm{sgn}(\frac{a}{b} + \frac{c}{d})$
$-$	$+$	$+$	$+$	$\mathrm{sgn}(x_0) = \mathrm{sgn}(\frac{a}{b} + \frac{c}{d})$
$-$	$-$	$+$	$-$	$\mathrm{sgn}(x_0) = \mathrm{sgn}(\frac{a}{b} + \frac{c}{d})$
$+$	$-$	$-$	$-$	$\mathrm{sgn}(x_0) = \mathrm{sgn}(\frac{a}{b} + \frac{c}{d})$

a	b	c	d	x_0
0	$+$	$+$	$+$	$+$
0	$+$	$-$	$+$	$-$
0	$-$	$+$	$-$	$-$
0	$-$	$-$	$-$	$+$
$+$	$+$	0	$+$	$+$
$-$	$+$	0	$+$	$-$
$-$	$-$	0	$-$	$+$
$+$	$-$	0	$-$	$-$

Condições necessárias e suficientes para existência de um único ciclo limite

Theorem.

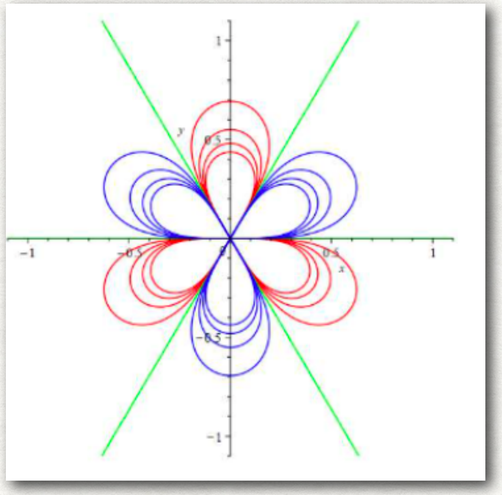
Para n=2,3,.... e a,b,d,x_0, y_0 satisfazendo as condições (*) temos que

$$\dot{z}^+ = i^m(z + z_0)^n \quad y > 0$$
$$\dot{z}^- = (a + bi)(z - d) \quad y < 0$$

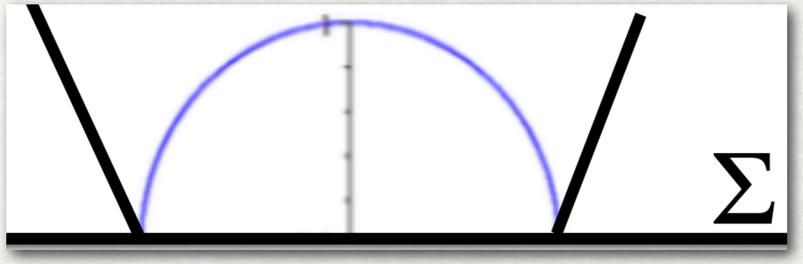
tem um único ciclo limites estável.

n par: m=0
n ímpar: m=1

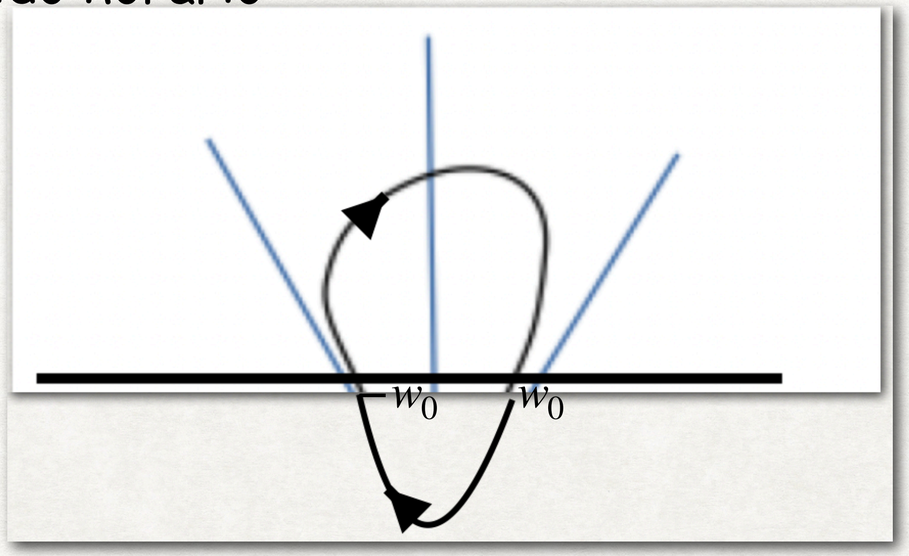
Prova. Vamos considerar casos dependendo da potência $n$do primeiro sistema $\dot{z}^+ = i^m(z+z_0)^n$. O retrato de fase de $\dot{z} = z^n$ tem $n-1$ raios que dividem o plano em $2(n-1)$ regiões.



Se n é par, uma das regiões contém o eixo imaginário. Neste caso consideramos $m=0, \quad z_0 = iy_0$ e então $\dot{z}^+ = (z+iy_0)^n$. Se n ié impar nós rodamos o retrato de fase multiplicando por i e então $\dot{z}^+ = i(z+iy_0)^n$. Assim em ambas situações o retro de fase no semi plano superior é como está mostrado na figura



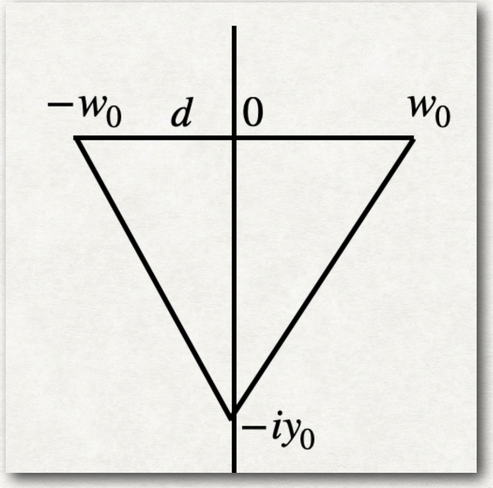
Considere $n = 4$. Para obtermos um ciclo limite consideramos um foco no semi plano superior . $\dot{z}^- = (a + bi)(z - d)$. Tomamos $a < 0, \quad b < 0$ e determinaremos d. O foco é atrator e gira no sentido horário



Para escolher d analisamos a aplicação de primeiro retorno. O retrato de fase de $\dot{z}^+ = (z + iy_0)^4$ é simétrico. Tomamos uma solução com condição inicial $z^+(0) = -w_0 < 0$ e a solução de $\dot{z}^- = (a + bi)(z - d)$ com condição inicial $z^-(0) = w_0$. Obtemos $z^-(\frac{-\pi}{b}) = d + (d - w_0)e^{-\frac{a\pi}{b}}$ e $\Pi(z) = d + (d + z)e^{-\frac{a\pi}{b}}$. O ponto fixo é $w_0 = \frac{d(1 + e^{\frac{a\pi}{b})})}{(1 - e^{\frac{a\pi}{b})})}$ com derivada $\Pi'(-w_0) = e^{-\frac{a\pi}{b}} < 1$.

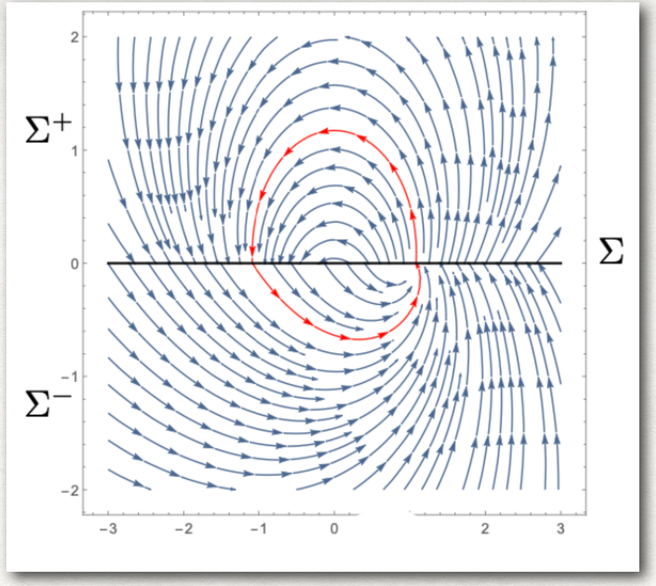
d<0 é escolhido tal que

$$\cot(\frac{2\pi}{3})y_0 < \frac{-d(1+e^{\frac{a\pi}{b})})}{(1-e^{\frac{a\pi}{b})})}$$

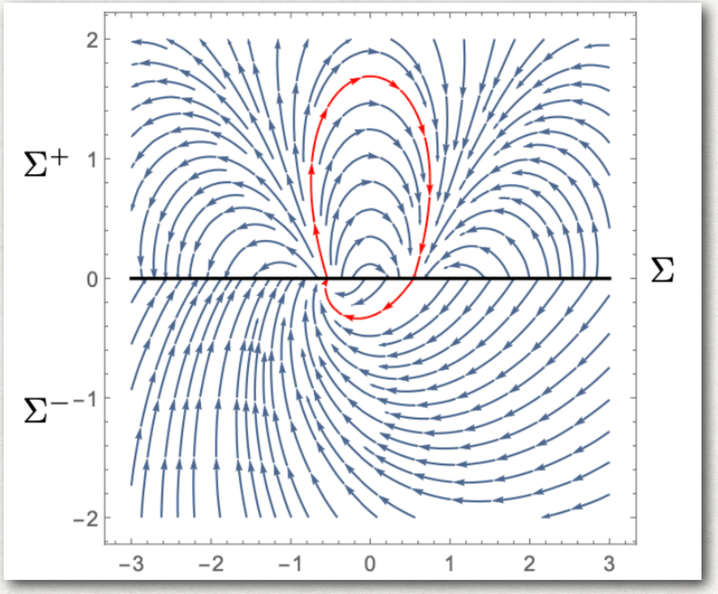


Para os demais casos a prova é similar e as condições são dadas por

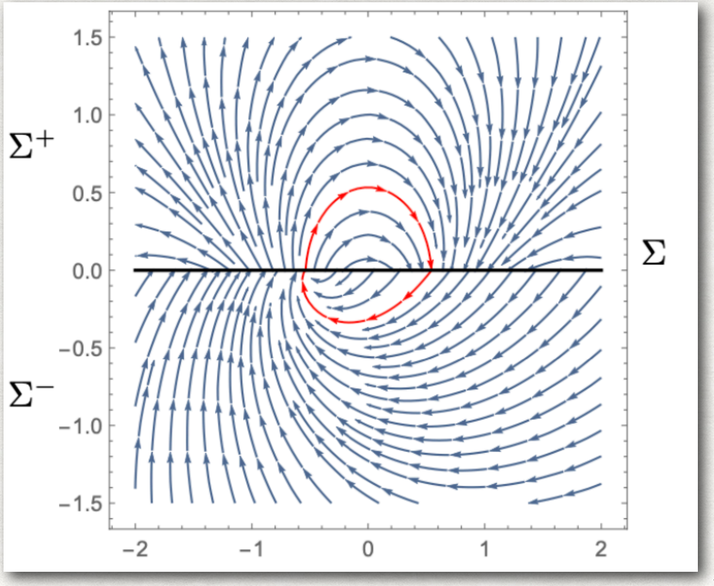
n	k	a	b	d	x_0	y_0	Main condition
2		$-$	$+$	$\mathbb{R}$	$\mathbb{R}$	$+$	$d > -x_0$
$4k-1$	$\geqslant 1$	$-$	$-$	$-$	0	$+$	$\cot\left(\frac{n\pi}{2(n-1)}\right) y_0 < -\frac{d(1+e^{\frac{a\pi}{b}})}{1-e^{\frac{a\pi}{b}}} < 0$
$4k$	$\geqslant 1$	$-$	$-$	$-$	0	$+$	$\cot\left(\frac{n\pi}{2(n-1)}\right) y_0 < -\frac{d(1+e^{\frac{a\pi}{b}})}{1-e^{\frac{a\pi}{b}}} < 0$
$4k-2$	>1	$-$	$+$	$+$	0	$+$	$0 < \frac{d(1+e^{\frac{a\pi}{b}})}{1-e^{\frac{a\pi}{b}}} < \cot\left(\frac{(n-2)\pi}{2(n-1)}\right) y_0$
$4k+1$	$\geqslant 1$	$-$	$+$	$+$	0	$+$	$0 < \frac{d(1+e^{\frac{a\pi}{b}})}{1-e^{\frac{a\pi}{b}}} < \cot\left(\frac{(n-2)\pi}{2(n-1)}\right) y_0$



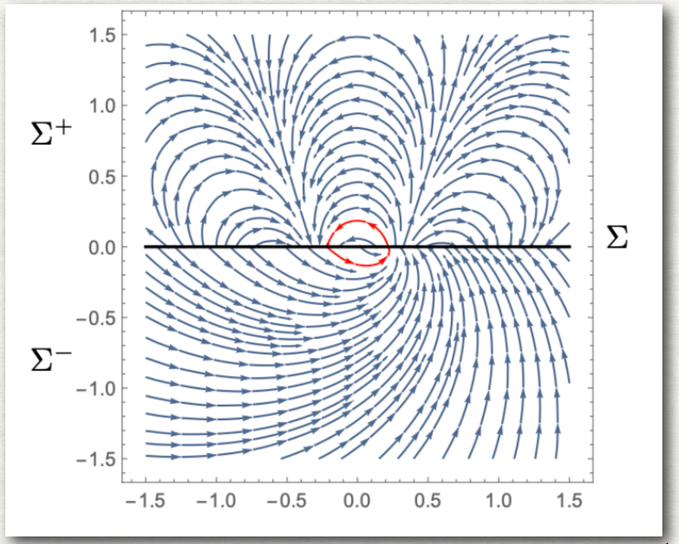
$$\dot{z}^+ = (z+i)^2, \quad \dot{z}^- = (-1+i)(z-1)$$



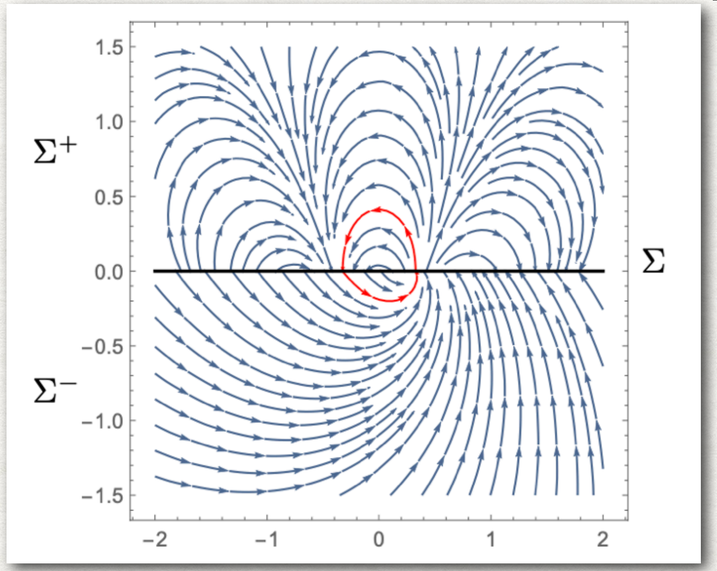
$$\dot{z}^+ = (z+i)^4, \quad \dot{z}^- = (-1-i)(z+\frac{1}{2})$$



$$\dot{z}^+ = i(z+i)^3, \quad \dot{z}^- = (-1-i)(z+\frac{1}{2})$$



$$\dot{z}^+ = (z+i)^6, \quad \dot{z}^- = (-1+i)(z-\frac{1}{5})$$



$$\dot{z}^+ = i(z+i)^5, \quad \dot{z}^- = (-1+i)(z-\frac{3}{10})$$

Teorema.

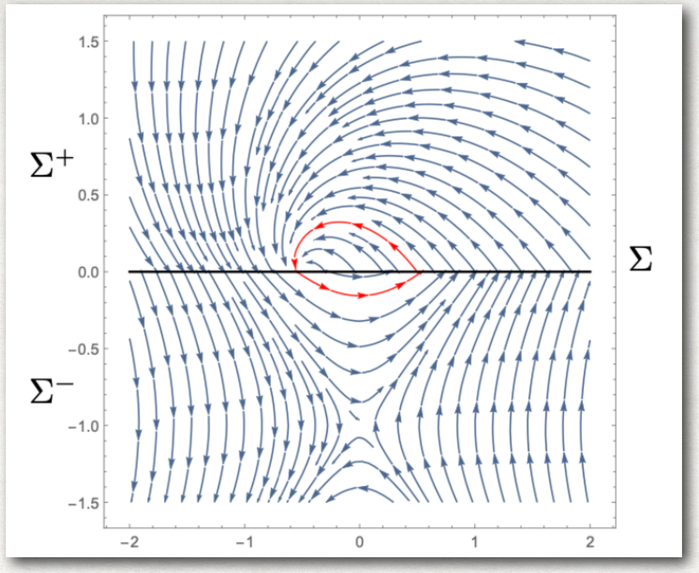
Para n=1,3,…. e a,b,d,y_0 reais não nulos satisfazendo (*)

$$\dot{z}^+ = (a+bi)(z-d) \quad y > 0$$

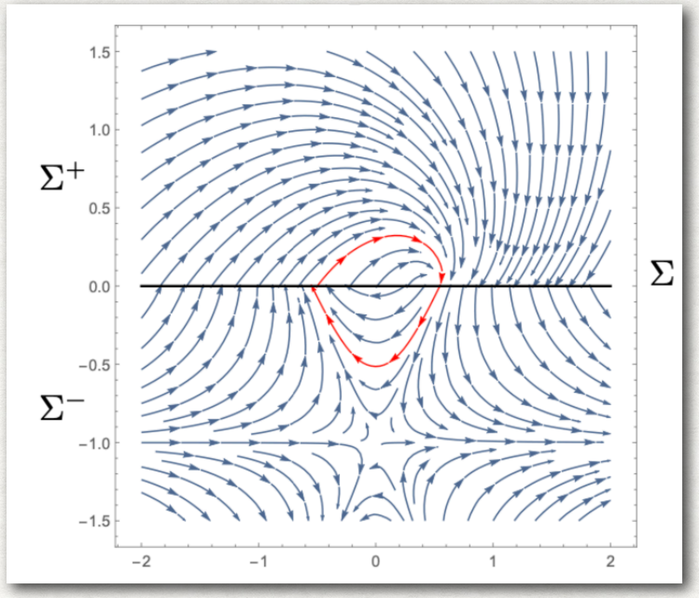
$$\dot{z}^+ = \frac{i^m}{(z+iy_0)^n} \quad y < 0$$

tem um único ciclo limite estável . (n par: m=0; n ímpar m=1)

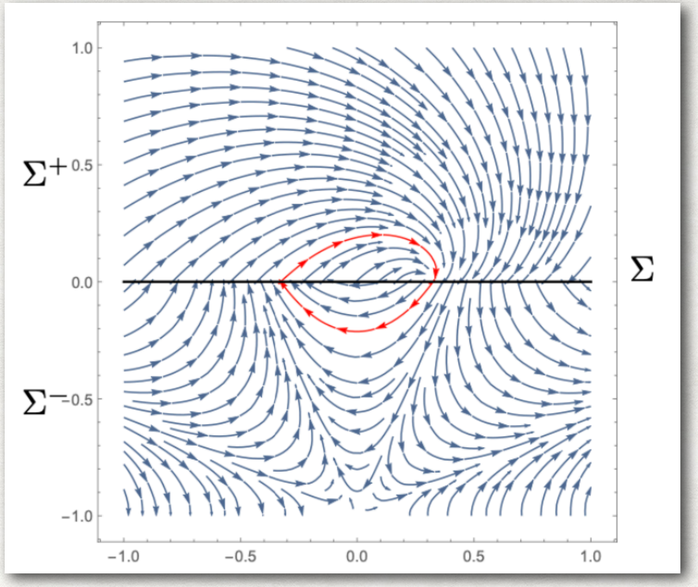
n	k	a	b	d	y_0	Main condition
$4k-2$	$\geqslant 1$	$-$	$-$	$+$	$+$	$0 < \frac{d(1+e^{\frac{a\pi}{b}})}{-1+e^{\frac{a\pi}{b}}} < \cot\left(\frac{n\pi}{2(n+1)}\right) y_0$
$4k-1$	$\geqslant 1$	$-$	$-$	$+$	$+$	$0 < \frac{d(1+e^{\frac{a\pi}{b}})}{-1+e^{\frac{a\pi}{b}}} < \cot\left(\frac{n\pi}{2(n+1)}\right) y_0$
$4k$	$\geqslant 1$	$-$	$+$	$-$	$+$	$\cot\left(\frac{(n+2)\pi}{2(n+1)}\right) y_0 < -\frac{d(1+e^{\frac{a\pi}{b}})}{-1+e^{\frac{a\pi}{b}}} < 0$
$4k+1$	$\geqslant 0$	$-$	$+$	$-$	$+$	$\cot\left(\frac{(n+2)\pi}{2(n+1)}\right) y_0 < -\frac{d(1+e^{\frac{a\pi}{b}})}{-1+e^{\frac{a\pi}{b}}} < 0$



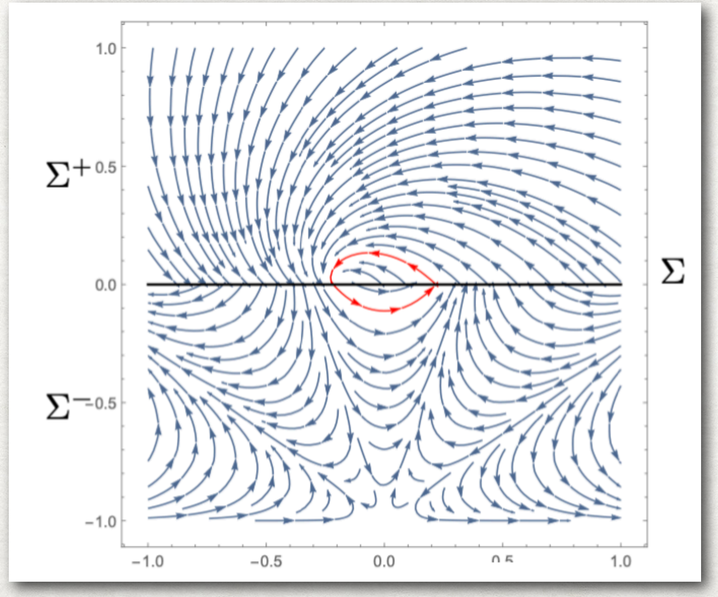
$$\dot{z}^+ = (-1+i)(z+\frac{1}{2}), \quad \dot{z}^- = \frac{i}{z+i}$$



$$\dot{z}^+ = (-1-i)(z-\frac{1}{2}), \quad \dot{z}^- = \frac{1}{(z+i)^2}$$



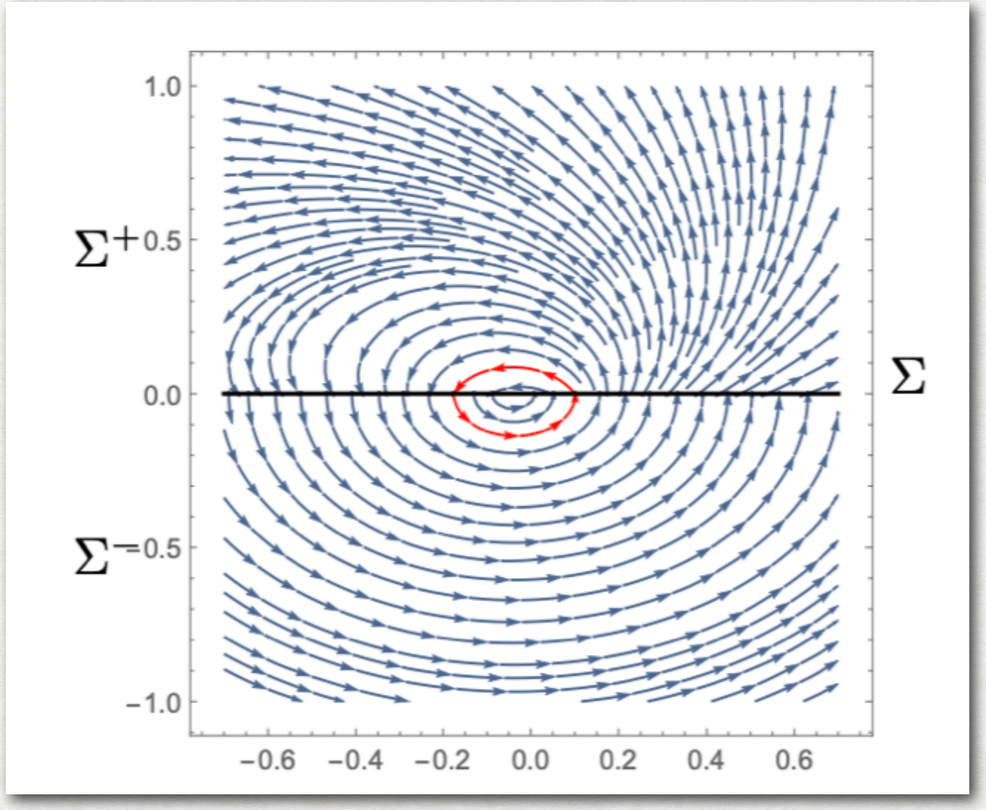
$$\dot{z}^+ = (-1-i)(z-\frac{3}{10}), \quad \dot{z}^- = \frac{i}{(z+i)^3}$$



$$\dot{z}^+ = (-1+i)(z+\frac{1}{5}), \quad \dot{z}^- = \frac{1}{(z+i)^4}$$

Example.

$$\dot{z}^+ = \frac{(z+\frac{i}{5})^2}{1+(z+\frac{i}{5})} \qquad \dot{z}^- = i(z+0.0381415)$$



Obtemos w_1, w_2 tais que

$\Delta(w_1) = \pi(w_1) - w_1 > 0,$

$\Delta(w_2) = \pi(w_2) - w_2 < 0,$

e concluímos que

existe $w_0 \in (w_1, w_2), \pi(w_0) = w_0$

Referência Básica:

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 332 (2022) 440–472

Journal of Differential Equations

www.elsevier.com/locate/jde

Piecewise holomorphic systems

Luiz F.S. Gouveia, Gabriel Rondón *, Paulo R. da Silva

São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences, Rua C. Colombo, 2265, CEP 15054-000, S. J. Rio Preto, São Paulo, Brazil

Received 19 January 2022; revised 24 May 2022; accepted 30 May 2022

Para uma primeira leitura :

ELSEVIER

Journal of Differential Equations

Volume 25, Issue 3, September 1977, Pages 310-324

Conformal equivalence of analytic flows

Louis Brickman, E.S Thomas

DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **17**, Number **2**, February **2007**

Website: http://aimSciences.org

pp. **309–329**

LOCAL AND GLOBAL PHASE PORTRAIT OF EQUATION $\dot{z} = f(z)$

ANTONIO GARIJO

Dep. d'Eng. Informàtica i Matemàtiques
Universitat Rovira i Virgili
Av. Països Catalans, 26, 43007 Tarragona, Spain

ARMENGOL GASULL

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellatera, Barcelona, Spain

XAVIER JARQUE

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona, Gran Via 585
08007 Barcelona, Spain