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LOCAL AND GLOBAL PHASE PORTRAIT OF EQUATION $\dot{z} = f(z)$

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Abstract. This paper studies the differential equation $\dot{z} = f(z)$, where f is an analytic function in $\mathbb{C}$ except, possibly, at isolated singularities. We give a unify treatment of well known results and provide new insight into the local normal forms and global properties of the solutions for this family of differential equations.

1. **Introduction.**

We consider the complex first order differential equation

$$\frac{dz}{dt} = f(z), \qquad z \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{1.1}$$

where f is an analytic function in $\mathbb{C}$ except, possibly, at isolated singularities. This is a rather general family of complex functions that includes polynomial, rational, entire and meromorphic functions, plus functions with isolated essential singularities.

The historical interest in this study has several sources. On the one hand, it is natural to wonder about the *limits* of the dynamics determined by the solutions of (1.1), such as curves in $\mathbb{C}$, because of the particularities of one-variable complex functions (the role of the Cauchy–Riemann equations in the dynamics). On the other hand, in we consider the classical theory of (real) differential equations and, more specifically, in Hilbert 16th problem, it is clear that the change of variables $z = x + iy$ transforms (1.1) into a system of planar ($\mathbb{R}^2$) differential equations

$$\begin{aligned} \dot{x} &= f_1(x, y), \\ \dot{y} &= f_2(x, y), \end{aligned} \tag{1.2}$$

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where $f_1(x,y) = \mathrm{Re}(f(x+iy))$ and $f_2(x,y) = \mathrm{Im}(f(x+iy))$. As we will show, system (1.2), though not a Hamiltonian system (unless the trivial case $f(z) = ikz,\ k \in \mathbb{R}$), has an integrating factor defined in the plane except some isolated points, given by the zeros of f and its essential singularities. Therefore, it mostly preserves a *Hamiltonian-like* behavior and it is natural to extend the well known *weak 16th Hilbert problem* to such a family, i.e. determine to how many limit cycles one can obtain in a general real perturbation (of the form $(\epsilon P(z,\bar{z}), \epsilon Q(z,\bar{z}))$) of system (1.2). With this key problem in mind, a first step is to obtain a good description of the global phase portrait of (1.1) in $\hat{\mathbb{C}}$.

Finally, we address another subject in which the study of the dynamics described by system (1.1) whenever $f(z)$ is a rational function of the form $f(z) = P(z)/Q(z)$ would be a powerful tool. The Euler method for numerically integrating system $\dot{z} = f(z)$ is given by $z_{n+1} = z_n + hf(z_n)$. Moreover, it is shown in [2] that there always exists a function g such that $P(z)/Q(z) = g/g'$. Of course, g is very general, presenting not only poles but essential singularities. Now, however, the Euler discrete system may be written as $z_{n+1} = z_n + h\ g(z_n)/g(z_n)$, which is the *relaxed* Newton method for the function g (when h is 1 we have the Newton method). Consequently, it is intuitively clear that the global phase portrait in $\mathbb{S}^2$ (the Riemann sphere) for system $\dot{z} = P(z)/Q(z)$ is closely related to the dynamics of the relaxed Newton method for the function g. This is an interesting subject in complex dynamics. In particular, it is shown in [2] that the Julia set and its dynamics is determined by the phase portrait of the rational system $\dot{z} = f(z)$ as long as h is small enough.

The above arguments demonstrate the interest in studying system (1.1) from the local and global points of view. We start by presenting and discussing our main results at the local level. Let us assume that f is an analytic function in a punctured neighborhood of $z = 0$, i.e. $z = 0$ is either a regular point, a singular point, a pole, or an essential singularity.

The classification of the local phase portrait around $z = 0$ is closely related to the characterization of the normal forms associated with the different classes of possible phase portrait. Roughly speaking, a normal form gives *the easiest* expression for all equations belonging to the same class and determines, at some level, the phase portraits of all equations in the class. *A priori* the normal form depends on the notion of *equivalence*, which defines the distinct classes, and is not necessarily unique since the word *easiest* may have different interpretations.

The general theory of dynamical systems has mostly been interested in the topological properties of the (local) phase portrait and the notion of topological equivalence has therefore been used to define the equivalence classes. Indeed, two systems defined in open sets U and V are topologically equivalent if there exists a homeomorphism from U to V that preserves the orbits but not necessarily the parametrization (time). Thus, for instance, two planar hyperbolic saddles are (locally) topologically equivalent and the natural normal form would be the linear system $(\dot{x} = x,\ \dot{y} = -y)$. This situation is no longer true, due to the existence of resonances, if in the notion of equivalence we require the preservation of the parametrization.

Using this classical notion of topological equivalence, some papers (see, for instance, [10],[11] and [20]) have discussed the characterization of the topological classes of system (1.1) in a neighborhood of an isolated singular point $z = 0$ (i.e. $f(0) = 0$) and reported that, for instance, there is no center-focus problem. In [10] and [15] it is also shown that if $z = 0$ is an isolated pole of order n of the function f

defining (1.1), its local phase portrait is topologically equivalent to a (generalized) saddle and its normal form is $\dot{z} = 1/z^n$.

We wish to explore here the significance of the fact that f is an analytic complex function in a punctured neighborhood of $z = 0$, in the notion of equivalence. Specifically, we want to determine whether the complex analyticity of f force, in a natural way, a much more restricted notion of equivalence where the homeomorphism is a holomorphic map. This direction has already been explored. In [3], it is shown that if $f(0) = 0$ and a technical condition on the residues is satisfied, the above homeomorphism is indeed a conformal map, so two topologically equivalent systems are conformally equivalent (the time is then holomorphically preserved). Our main result on local conformal conjugacy is the following:

Theorem 1.1. *Let $f(z)$ be an analytic function in a punctured neighborhood of p. By shrinking this neighborhood, if necessary, the corresponding equation $\dot{z} = f(z)$ is conformally conjugated, near $z = 0$, to*
(a) $\dot{z} = 1$, if $f(0) \neq 0$,
(b) $\dot{z} = f'(0)z$, if $z = 0$ is a zero of f of order 1 (i.e., $f'(0) \neq 0$),
(c) $\dot{z} = z^n - cz^{2n-1}$, where $c = Res(1/f, 0)$, if $z = 0$ is a zero of f of order $n > 1$,
(d) $\dot{z} = 1/z^n$, if $z = 0$ is a pole of order n.

The above theorem includes [3]'s result (cases (b) and (c)). Case (d) is also considered in [9] in a different framework. In any case, our proof uses a completely different approach known as the *homotopic method* (see [12] for details). The homotopic method uses Lie symmetries (see [16]) to determine the normal form for all cases, including the poles, in the same manner. Intuitively, the homotopic method involves finding a path in the space of the equations such as (1.1) such that all are conformally conjugated. In one extreme of the path we would have, for instance, equation (1.1) with any function f such that $f(0) = 0$ and $f'(0) \neq 0$, and on the other extreme of the path we would have the equation (1.1) given by $\dot{z} = f'(0)z$. Another proof of Theorem 1.1, based on the explicit search of the conjugacy, can be found in [7].

Note that the normal form proposed in [3] for item (c) is given by $\dot{z} = z^n/(1 + cz^{n-1})$. Alternatively, $\dot{z} = z^n - c^{1/(n-1)}z^{n+1}$ can be used.

Using mainly Theorem 1.1, we also prove the following result for the conformal conjugacy classes in $\mathbb{C} \setminus \overline{\mathbb{D}(0, R)}, R >> 1$, i.e., in a sufficiently small neighborhood of infinity. At the topological level, the study of the phase portrait at infinity was done in [15].

Theorem 1.2. *Let $f(z)$ be a rational function, i.e., $f(z) = P(z)/Q(z)$, where $P(z) = a_n z^n + \ldots + a_0$ and $Q(z) = b_m z^m + \ldots + b_0$ are polynomials in z of degree n and m, respectively. Let c be the residue of $g(z) = -\frac{Q(1/z)}{z^2 P(1/z)}$ at $z = 0$. Then, there exists $R > 0$ such that the corresponding equation $\dot{z} = f(z)$ is conformally conjugated, in $\mathbb{C} \setminus \overline{\mathbb{D}(0, R)}$, to*

(a) $\dot{z} = (1/z)^{m-n} + c\,(1/z)^{2(m-n)+1}$, if $n < m + 1$,
(b) $\dot{z} = (a_n/b_m)\, z$, if $n = m + 1$,
(c) $\dot{z} = 1$, if $n = m + 2$,
(d) $\dot{z} = z^{n-m}$, if $n > m + 2$.

The above theorem deals only with rational equations rather than more general functions because we do not want infinity to be an essential singularity. Indeed, we do not completely understand the equivalent classes (at the topological as well as the conformal level) of phase portraits near some simple essential singularities. However, to show the topological phase portrait around some essential singularities, we study the local phase portrait at $z = 0$ for the family of differential equations $\dot{z} = z^m \exp(1/z^n)$, where $n, m \in \mathbb{N}$.

As usual, after studying the local phase portrait near a regular or singular point, the next step is to determine the topological behavior of the orbits in a neighborhood of a periodic orbit. In this sense, we may say that we are dealing with global properties of the solutions of equation (1.1). It has been repeatedly (and independently) proven that equation (1.1) under the analytic, rational or meromorphic framework may not present limit cycles (isolated periodic orbits). Moreover, if C is a periodic orbit, then all orbits nearby are periodic and have the same period. The case of f (in equation (1.1)) being analytic can be found in [4, 6, 20], while the rational or meromorphic case is treated in [2, 10, 11, 15].

We further develop this study by showing that the *Hamiltonian-like* behavior of the differential equation (1.1) is due to the existence of an integrating factor that is well defined everywhere except at the zeros and essential singularities of f. So, the almost everywhere preserving area property is compatible with the existence of focus and nodes. To prove this, we again use the existence of a Lie symmetry. Of course, from this observation, as a corollary we get the non-existence of limit cycles and able to extend this *preserving area phenomenon* to a neighborhood of any monodromic graph, i.e. a graph for which some return map can be defined. We summarize all these implications in the following theorem.

Theorem 1.3. *Let equation (1.1). The following statements hold.*

(a) There are no limit cycles. Moreover, in any neighborhood of a periodic orbit C in which the return map is defined, all orbits are periodic and have the same period as C. Furthermore, this period is

$$T = \int_C \frac{1}{f(z)}\, dz.$$

(b) If Γ is a monodromic graph, then in any ring-shape neighborhood of Γ in which the return map is defined, all orbits of the differential equation are periodic.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and Theorem 1.2. We also discus the phase portrait around an essential singularity. In Section 3 we show the Hamiltonian structure of equation (1.1) and prove Theorem 1.3. Finally, in Section 4, we give a topological classification of the phase portraits in the Poincaré disc of all rational differential equations $\dot{z} = P(z)/Q(z)$ such that $\deg(P) \leq 2$ and $\deg(Q) \leq 1$ without common factors.

2. **Local normal forms.** The aim here is to study the (local) dynamics of equation (1.1) in some punctured neighborhood of a point p that, without loss of generality, we assume is the origin, $z = 0$. In the usual way, at $z = 0$, the map f is either regular or singular. Moreover, in this second case, it can have a *zero* ($f(z) = 0$), or a *singularity* (f is not well defined at $z = 0$) that is either a pole or an essential singularity.

Before proving the main results of this section, Theorem 1.1 (Section 2.1) and Theorem 1.2 (Section 2.3), we define the notion of conformal conjugacy and state well-known results that we will use later. For a given z close to the origin, we denote by $\varphi_f(t, z)$ the solution of the equation (1.1) passing through z at $t = 0$.

We say that $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are *conformally conjugated near the origin* if there exists a conformal function $\Phi : U \to V$, where U and V are two open neighborhoods of the origin, such that $\Phi(0) = 0$ and

$$\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z)), \quad \text{for all} \quad z \in U \setminus \{0\}, \tag{2.3}$$

and for all t for which the above expressions are well defined and the corresponding points are in U and V.

Remark 2.1. The notion of conformal conjugacy is, of course, a stronger notion than topological conjugacy or topological equivalency. In particular, under conformal conjugacy the angles in the tangent space are preserved.

The following lemmas, first proven in [3], give a useful characterization and establish two necessary conditions for equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ being conformally conjugated near the origin.

Lemma 2.2. *The equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated near the origin if and only if there exists a conformal function $\Phi : U \to V$, where U and V are two open neighborhoods of the origin, such that $\Phi(0) = 0$ and*

$$\Phi'(z) f(z) \;=\; g(\Phi(z)) \qquad \textit{for all} \;\; z \in U \setminus \{0\}.$$

Proof. This is straightforward if we take the derivative with respect to t in equation (2.3). □

Lemma 2.3. *Let $f(z)$ and $g(z)$ be two analytic functions in some punctured neighborhood of the origin. If the corresponding equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated near the origin, then*

(a) $Res(1/f, 0) = Res(1/g, 0)$ and,

(b) the order of vanishing of f and g at the origin coincide, if one is finite.

Proof. Let $\Phi : U \to V$ be the conformal conjugacy. Denote by γ a small circle around the origin contained in U. We then have

$$2\pi i Res(1/g, 0) \;=\; \int_{\Phi(\gamma)} \frac{dw}{g(w)} \;=\; \int_{\gamma} \frac{\Phi'(z)\, dz}{g(\Phi(z))} \;=\; \int_{\gamma} \frac{dz}{f(z)} = 2\pi i Res(1/f, 0),$$

where the third equality follows from Lemma 2.2. This proves (a).

To see the equal order of vanishing at the origin we just check the orders in equation $\Phi'(z) f(z) = g(\Phi(z))$. If Φ exists and is conformal, we have $\Phi(z) = O(z)$ and $\Phi'(z) = O(1)$. Thus, the orders of vanishing of f and g, if finite, must be equal. If both are infinite, i.e. $z = 0$ is an essential singularity, we can derive a similar condition (see Section 2.4). □

2.1. **Proof of Theorem 1.1.** We use here what is known as the *homotopic method* (see [12]), which is essentially based on the properties of the Lie symmetries. Independent proofs of this result can be found in [7],[3] (except statement (d)) and [9] (statement (d)).

For the sake of completeness, we will first briefly introduce some properties of the Lie brackets and show, as a lemma, a suitable application that we will use later. Let

$$X = \left\{ \begin{array}{l} \dot{z} = X_1(z, s), \\ \dot{s} = X_2(z, s), \end{array} \right.$$

and

$$Y = \begin{cases} \dot{z} = Y_1(z,s), \\ \dot{s} = Y_2(z,s), \end{cases}$$

be two systems of complex differential equations in $\mathbb{C}^2$. Denote by $\varphi(t; z_0, s_0)$ and $\phi(\tau; z_0, s_0)$ the solutions of systems X and Y satisfying $\varphi(0; z_0, s_0) = (z_0, s_0)$ and $\phi(0; z_0, s_0) = (z_0, s_0)$, respectively. We say that systems X and Y *commute* if $\phi(\tau; \varphi(t; z_0, s_0)) = \varphi(t; \phi(\tau; z_0, s_0))$. To obtain a useful tool for deciding whether the systems commute, the Lie bracket $[X, Y]$ is introduced as follows

$$[X,Y] = (DX)\,Y - (DY)\,X, \tag{2.4}$$

where D denotes the differential matrix. Specifically, it can be shown that two systems commute if and only if their corresponding Lie bracket vanishes (see, for instance, [16] for an overview of this subject).

Lemma 2.4. *Let u and v be two analytic functions in a punctured neighborhood of the origin such that $v(z) = o(u(z))$ (i.e., $\lim_{z\to 0} v(z)/u(z) = 0$). Assume that*

$$h(z,s) := (u(z) + sv(z)) \int_0^z \frac{v(\xi)}{(u(\xi)+sv(\xi))^2} d\xi \tag{2.5}$$

defines, for all $s \in [0,1]$, a holomorphic function in a whole neighborhood of the origin (including $z = 0$) and that $h(0,s) \equiv 0$. Then the equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$ are conformally conjugated near the origin.

Proof. In the assumptions of the lemma we consider the equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$. Moreover, we also build system X as

$$\begin{aligned} \dot{z} &= u(z) + sv(z), \\ \dot{s} &= 0, \end{aligned} \tag{2.6}$$

where $s \in [0,1]$, and system Y as

$$\begin{aligned} \dot{z} &= H(z,s), \\ \dot{s} &= 1. \end{aligned} \tag{2.7}$$

By definition, if we find a holomorphic function H for system Y such that systems X and Y commute, then we would have $\phi(\tau; \varphi(t; z_0, s_0)) = \varphi(t; \phi(\tau; z_0, s_0))$. Consequently, by taking $\tau = 1$ and $s_0 = 0$, we would have $\phi(1; \varphi(t; z_0, 0)) = \varphi(t; \phi(1; z_0, 0))$. This latter equality is precisely the condition for the equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$ to be holomorphically conjugated. Indeed, the holomorphic map (Φ in the definition) that conjugates both equations is given by the holomorphic solutions of system Y at "time" $\tau = s = 1$ (see Figure 1).

We now show that under condition (2.5) there exists a holomorphic function H defined in the whole neighborhood of the origin such that $[X, Y] = 0$. From (2.4) we know that

$$[X,Y] = \begin{pmatrix} u_z + sv_z & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} H \\ 1 \end{pmatrix} - \begin{pmatrix} H_z & H_s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u + sv \\ 0 \end{pmatrix}$$

Clearly, $[X, Y] = 0$ rises into the first order linear equation $(u_z + sv_z)H + v - H_z(u + sv) = 0$. Clearly, one of its solutions is $H(z,s) = h(z,s)$, where h is the holomorphic function given in the statement of the lemma. The fact that $h(0,s) \equiv 0$ implies that the solution $z = 0$ of (2.7) is defined for all $\tau \in \mathbb{R}$ and, as a consequence, the flow ϕ of (2.7) in a neighborhood of the initial condition $z = 0$ is well defined for $\tau = 1$, as needed. ∎

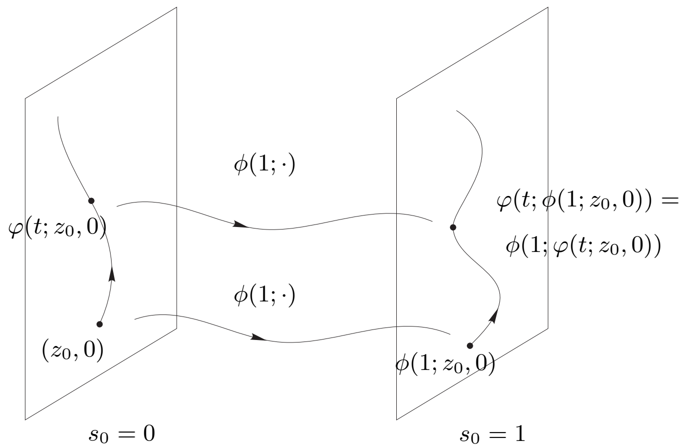

FIGURE 1. The solutions of system (2.7) conjugate equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$.

Proof of Theorem 1.1. We consider each case separately.

Proof of (b). If we write $u(z) = f'(0)z$ and $v(z) = O(z^2)$, statement (b) is equivalent to showing that equation $\dot{z} = u(z)$ is conformally conjugated to equation $\dot{z} = u(z) + v(z)$. But this is the case using Lemma 2.4, since we have that

$$h(z,s) = (u(z) + sv(z)) \int_0^z \frac{v(\xi)}{(u(\xi) + sv(\xi))^2} d\xi = (f'(0)z + sO(z^2)) \int_0^z \frac{O(\xi^2)}{(f'(0)\xi + sO(\xi^2))^2} d\xi$$

defines a holomorphic map around $z = 0$ for all $s \in [0,1]$ and $h(0,s) \equiv 0$.

Proof of (c). This case follows in three steps. First we notice that equation $\dot{z} = f(z) = az^n + a_1 z^m + O(z^{m+1}),\ a \neq 0,\ m > n$ is conformally conjugated to equation $\dot{w} = w^n + b_1 w^m + O(w^{m+1})$ with $b_1 = a_1 \sqrt[n-1]{1/(a^{m-1})}$ via the linear change of variable $z = \sqrt[n-1]{1/a}\ w$. Second, renaming the equation into the $z-$variable, we claim that equation $\dot{z} = z^n + a_m z^m + O(z^{m+1}),\ n < m < 2n-1$ is conformally conjugated to equation $\dot{w} = w^n - cw^{2n-1} + O(w^{2n})$, where c is $\mathrm{Res}(1/f, 0)$. To see the claim, we consecutively apply the change of variable $z = w(1 + \alpha w^{m-n})$, where $\alpha = -a_m/(2n-m-1)$ to erase the mth degree monomial up to $m = 2n-2$. The monomial w^{2n-1} cannot be erased since it is a resonance term. Moreover, using Lemma 2.3 it is easy to prove that the coefficient of this monomial coincides with c. Hence, again renaming the equation into the $z-$variable, we assume that the initial equation becomes $\dot{z} = z^n - cz^{2n-1} + O(z^{2n})$. Third, we show that equation $\dot{z} = z^n - cz^{2n-1} + O(z^{2n})$ is conformally conjugated to equation $\dot{z} = z^n - cz^{2n-1}$. To do so, we apply Lemma 2.4, with $u(z) = z^n - cz^{2n-1}$ and $v(z) = O(z^{2n})$. We have

$$\begin{aligned} h(z,s) &= z^n \left(1 - cz^{n-1} + sO(z^n)\right) \int_0^z \frac{\xi^{2n}(1 + O(\xi^{n+1}))}{\xi^{2n}\left(1 - c\xi^{n-1} + sO(\xi^n)\right)^2} \\ &= z^n \left(1 - cz^{n-1} + sO(z^n)\right) \int_0^z \frac{1 + O(\xi^{n+1})}{\left(1 - c\xi^{n-1} + sO(\xi^n)\right)^2} d\xi, \end{aligned} \tag{2.8}$$

which defines a holomorphic function, for all $s \in [0,1]$, in a neighborhood of the origin. Again $h(0,s) \equiv 0$ and (c) follows.

Proof of (a) and (d). To unify the regular case and the case when f has a pole, we consider $n \in \mathbb{N} \cup \{0\}$. As in the previous case, it is easy to see that equation $\dot{z} = f(z) = az^{-n} + O\left(z^{1-n}\right)$ is conformally conjugated to equation $\dot{z} = f(z) = z^{-n} + O\left(z^{1-n}\right)$ via a linear change of variable. We claim that equation $\dot{z} = z^{-n} + O\left(z^{1-n}\right)$ is conformally conjugated to equation $\dot{z} = z^{-n}$. To prove this claim, we use Lemma 2.4, with $u(z) = z^{-n}$ and $v(z) = O\left(z^{1-n}\right)$. Precisely, equation (2.5) becomes

$$h(z,s) = \left(z^{-n} + sO\left(z^{1-n}\right)\right) \int_0^z \frac{O\left(\xi^{1-n}\right)}{\left(\xi^{-n} + O\left(\xi^{1-n}\right)\right)^2}\, d\xi = \left(z^{-n} + sO\left(z^{1-n}\right)\right) \int_0^z O(\xi^{n+1})\, d\xi,$$

which implies that $h(z,s) = O(z^2)$ is holomorphic in a sufficiently small neighborhood of the origin and $h(0,s) \equiv 0$. □

Remark 2.5. As Theorem 1.1 shows, the only case in which the normal form is not given by the "first" term of Taylor's series defining the equation $\dot{z} = f(z)$ at $z = 0$ corresponds to the analytic case with $f'(0) = 0$. However, if only topological equivalence is required (instead of conformal conjugacy) we claim that the normal form of $\dot{z} = z^n + O(z^{n+1})$ becomes simply $\dot{z} = z^n$. To prove this claim, we can use either the blow-up technique, or the Lie bracket properties, as in the proof of Lemma 2.4, but instead to ask for $[X,Y] = 0$ (i.e., X and Y commutes) we look for an analytic system Y such that $[X,Y] = \mu X$ for some well-defined function $\mu(z,s) \neq 0$. In this case, the solution curves of Y "send" orbits of X to orbits of X but does not preserve the time parametrization. Of course the solution curves of Y would give the equivalency between $\dot{z} = z^n + O(z^{n+1})$ (indeed we may take $\dot{z} = z^n + cz^{2n-1}$, because of Theorem 1.1) and $\dot{z} = z^n$. To show the existence of such a system Y, we argue, as in the proof Lemma 2.4, that $u(z) = z^n$ and $v(z) = cz^{2n-1}$. Thus, we wish to find an analytic function H (and the corresponding function $\mu(z,s)$) such that $[X,Y] = \mu X$. Similar manipulations, as in the proof of Lemma 2.4, give rise to the expression

$$H(z,s) = (z^n + scz^{2n-1}) \int_0^z \frac{c\xi^{2n-1} - \mu(\xi,s)(\xi^n + sc\xi^{2n-1})}{(\xi^n + sc\xi^{2n-1})^2}\, d\xi. \tag{2.9}$$

So, if we choose $\mu(z,s) = z^{n-1}(c-z)/\left(1 + scz^{n-1}\right)$, then the function $H(z,s)$ becomes

$$H(z,s) = (z^n + scz^{2n-1}) \int_0^z \frac{1}{(1 + sc\xi^{n-1})^2}\, d\xi\ , \tag{2.10}$$

which is an analytic function satisfying $H(0,s) \equiv 0$ and the claim follows.

2.2. **Consequences of Theorem 1.1 and local phase portraits.** In this subsection we use Theorem 1.1 to obtain the phase portraits of $\dot{z} = f(z)$ in a punctured neighborhood of a singular point ($f(0) = 0$) or a singularity given by a pole.

Corollary 2.6. *Let $f(z)$ and $g(z)$ be two analytic functions in some punctured neighborhood of the origin. The corresponding differential equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated if and only if they satisfy (a) and (b) of Lemma 2.2.*

Proof. The proof is immediate using the normal forms of Theorem 1.1. □

Corollary 2.7. *Let $f(z)$ be an analytic function in a punctured neigbourhood of p.*

(a) *If $f'(p) \neq 0$, according to $Re(f'(p)) < 0$, $Re(f'(p)) > 0$ or $Re(f'(p)) = 0$, then the phase portrait of equation (1.1) in a neigbourhood of p is a stable focus, an unstable focus or an isochronous center, respectively. In all cases, the index of p is 1.*

(b) *If p is a zero of f of order $n > 1$, then the phase portrait of equation (1.1) in a neigbourhood of p is a union of $2(n-1)$ elliptic sectors, and so the index of p is n.*

(c) *If p is a pole of f of order $n \geq 1$, the phase portrait of equation (1.1) in a neigbourhood of p is a union of $2(n+1)$ hyperbolic sectors, and so the index of p is $-n$.*

Proof. Statement (a) follows from Theorem 1.1(b), since equation (1.1) near p is conformally conjugated to its linear part, $\dot{z} = f'(p)z$.

Statements (b) and (c) follow by performing the polar blow-up $z = re^{i\theta}$ and a recalling of time. □

Remark 2.8. The following two considerations hold.

(i) Because of the conformal conjugacy, we know that the aperture of all the sectors (elliptic as well as hyperbolic) of the phase portrait at $z = p$ must be equal.

(ii) Notice that the phase portrait near $z = p$, where p is a zero or a pole of equation (1.1), is totally determined by its index, except when the index is 1 (where we can have centers, focus or nodes). This is a very unusual property which holds due to the rigid structure of the differential equation.

2.3. **The dynamics at infinity.** To better understand the *global* phase portrait of equation (1.1) we need to study the behavior of the orbits in a neighborhood of infinity, i.e., we need to know how the orbits escape to infinity, if indeed they do. The main result is summarized in Theorem 1.2.

In the context of planar (real) vector fields, there are two natural compactifications that allow us to extend the flow to "infinity" under certain hypotheses. These are the Riemann compactification, where infinity is represented by the north pole of the *Riemann sphere*, and the Poincaré compactification, where infinity is represented by the equator of the *Poincaré sphere*, which is invariant by the new flow. In this second case, we have two copies of the flow, one on each hemisphere. By direct projection, the vector field defined in the north hemisphere of the *Poincaré sphere* can be drawn in the so-called *Poincaré disc*, $\mathbb{D}^2$, where infinity now is given by its boundary $\mathbb{S}^1$. Indeed the $\mathbb{S}^1$ of the Poincaré disc corresponds to the blow up of the north pole of the Riemann sphere. See [13] for details and precise equations for both compactifications.

Here, we can consider a similar approach but taking into account the complex structure of equation (1.1). It is easy to see that infinity can be formally interpreted by $z = \infty$ (the north pole of the Riemann sphere), so it can be directly studied by considering the change of variables $w = 1/z$. Thus, $z = \infty$ for equation (1.1) becomes $w = 0$ in the new variable. To study the dynamics (and normal forms) around $w = 0$, we use Theorem 1.1. Finally, using the inverse change of variables $z = 1/w$, we will find the normal form of the original equation (1.1) in a neighborhood of infinity.

Proof of Theorem 1.2: Assume that f for system (1.1) is a rational function, i.e.,

$$\dot{z} = \frac{P(z)}{Q(z)} = \frac{a_n z^n + \cdots + a_0}{b_m z^m + \cdots + b_0} \text{ with } a_n b_m \neq 0. \tag{2.11}$$

After doing the change of variables $w = 1/z$ we have

$$\dot{w} = -w^2 \frac{P(1/w)}{Q(1/w)} = -w^{2+m-n} \frac{a_n + \cdots + a_0 w^n}{b_m + \cdots + b_0 w^m} = -w^{2+m-n}(\frac{a_n}{b_m} + \mathcal{O}(w)). \quad (2.12)$$

Clearly, the dynamics of equation (2.12) in a neighborhood of $w = 0$ represents the dynamics of equation (2.11) in a neighborhood of $z = \infty$. From Theorem 1.1 we know that, in some open neighborhood, say $D(0, \epsilon_0) = \{w \in \mathbb{C}, |w| < \epsilon_0\}$, of $w = 0$, the normal form for equation (2.12) is given, respectively, by,

(a) $\dot{w} = w^{2+m-n} + cw^{2(2+m-n)-1}$, if $n < m + 1$,
(b) $\dot{w} = (a_n/b_m)\, w$, if $n = m + 1$,
(c) $\dot{w} = 1$, if $n = m + 2$,
(d) $\dot{w} = w^{-n+m+2}$, if $n > m + 2$,

where c is the residue at the origin of the function $g(w) = -\frac{Q(1/w)}{w^2 P(1/w)}$. We know that there is a a conformal conjugacy Φ between (2.12) and its corresponding normal form.

We now get back to the original z-variable by applying the inverse change of variables $z = 1/w$ to the above normal forms. Consequently, in a suitable neighborhood of infinity (precisely, $\mathbb{C} \setminus \overline{D(0, R)}$, with $R = 1/\epsilon_0$), the normal form of equation (2.11) becomes

(a) $\dot{\tilde{z}} = -(\tilde{z})^{n-m} - c\,(\tilde{z})^{2(n-m)-1}$, if $n < m + 1$,
(b) $\dot{\tilde{z}} = (a_n/b_m)\, \tilde{z}$, if $n = m + 1$,
(c) $\dot{\tilde{z}} = -\tilde{z}^2$, if $n = m + 2$,
(d) $\dot{\tilde{z}} = -\tilde{z}^{n-m}$, if $n > m + 2$,

Furthermore, the linear change of variable $z = \lambda\tilde{z}$ (with $\lambda = (-1)^{1/(m-n+1)}$, $\lambda = 1$, $\lambda = -1$, and $\lambda = (-1)^{1/(1-m+n)}$ depending on case (a),(b),(c) and (d)) gives the normal forms stated in the theorem.

Finally, we observe that the above normal forms and the original equation (2.11) are conformally conjugated on $\mathbb{C} \setminus \overline{D(0, R)}$ under the map $H(z) = \lambda/\Phi(1/z)$. □

To end this section we particularize the above result for a concrete family of rational systems for which we will study, in Section 4, their global phase portrait.

Corollary 2.9. *Consider equation (2.11), where $P(z)$ and $Q(z)$ are polynomials of one complex variable with $\deg(P) \le 2$ and $\deg(Q) \le 1$. Its phase portrait in a neighborhood of infinity is conformally conjugated to the phase portrait in a neighborhood of infinity of one of the following normal forms*

(a) $\dot{z} = 1$ *if* $\deg(P) = 0$ *and* $\deg(Q) = 0$. *The topological phase portrait in a neighborhood of infinity is given by Figure 2(i).*
(b) $\dot{z} = kz$ *if* $\deg(P) = 1$ *and* $\deg(Q) = 0$, *or,* $\deg(P) = 2$ *and* $\deg(Q) = 1$. *The topological phase portrait in a neighborhood of infinity is given, modulus the orientation of the orbits, by Figure 2((ii)-(iv)) depending on* $k = \alpha$, $k = i\beta$ *or* $k = \alpha + i\beta$, *respectively.*
(c) $\dot{z} = z^2$ *if* $\deg(P) = 2$ *and* $\deg(Q) = 0$. *The topological phase portrait in a neighborhood of infinity is given by Figure 2(v).*
(d) $\dot{z} = 1 + c/z$ *if* $\deg(P) = 1$ *and* $\deg(Q) = 1$. *The topological phase portrait in a neighborhood of infinity is given by Figure 2(i).*
(e) $\dot{z} = 1/z + c/z^3$ *if* $\deg(P) = 0$ *and* $\deg(Q) = 1$. *The topological phase portrait in a neighborhood of infinity is given by Figure 2(vi).*

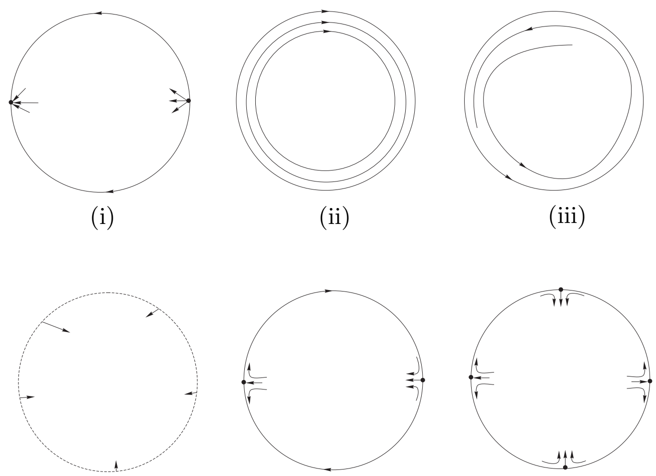

FIGURE 2. Phase portrait of equation (1.1) near infinity according to Corollary 2.9.

2.4. **The essential singularity case.** In this section we consider equation (1.1) with f having an isolated essential singularity at $z = 0$. We have not found a full characterization theorem for this kind of singularities, but we can extend the necessary condition given in statement (b) of Lemma 2.3 (statement (a) also works with essential singularities) when the order of vanishing of the two equations is infinite.

Let f be a holomorphic map with an isolated essential singularity at $z = 0$. We denote by $M(\epsilon, f) = \max_{|z|=\epsilon}\{|f(z)|\}$, and define the *growth order* of f at $z = 0$ as

$$\rho_2(f) = \limsup_{\epsilon\to 0} \frac{\log(\log(M(\epsilon, f)))}{-\log(\epsilon)}.$$

It is easy to check that $\rho_2(\exp(1/z^n)) = n,\ n \geq 1$, while $\rho_2(\exp(\exp(1/z))) = \infty$. In the latter case we need to define

$$\rho_3(f) = \limsup_{\epsilon\to 0} \frac{\log(\log(\log(M(\epsilon, f))))}{-\log(\epsilon)}$$

thus generalizing the definition of $\rho_2(f)$. Inductively, we can define $\rho_k(f)$ for any natural index $k \geq 2$, in the case that $\rho_{k-1}(f) = \infty$.

Lemma 2.10. *Let f and g be two holomorphic functions with an isolated essential singularity at $z = 0$. Suppose that $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated near $z = 0$. If there exists $k \geq 2$ such that $\rho_k(f) < +\infty$, then $\rho_k(f) = \rho_k(g)$.*

Proof. An equivalent condition (Lemma 2.2) for the conformal conjugacy between $\dot{z} = f(z)$ and $\dot{z} = g(z)$ is the existence of a conformal map Φ with $\Phi(0) = 0$, such that

$$\Phi'(z)f(z) = g(\Phi(z)).$$

By studying the growth order in the above equation and using $\Phi'(z) = \mathcal{O}(1)$ and $\Phi(z) = \mathcal{O}(z)$, we obtain the desired result. ∎

2.4.1. *The family* $\dot{z} = z^m \exp(1/z^n)$. We study the local phase portrait at $z = 0$ for the family of equations

$$\dot{z} = z^m \exp(1/z^n), \tag{2.13}$$

where $n \geq 1$ and $m \geq n+1$. Of course, the origin is always an essential singularity and the growth order is n. As a consequence of Lemma 2.10, note that each value of n gives rise to a different class of conformal conjugacy.

After a scaling of the time and writing $z = x + iy$, system (2.13) becomes a real smooth planar system in a punctured neighborhood of the origin of the form

$$\begin{cases} \dot{x} = & P_m(x,y)\cos(\frac{R_n(x,y)}{(x^2+y^2)^n}) - Q_m(x,y)\sin(\frac{R_n(x,y)}{(x^2+y^2)^n}) \\ \dot{y} = & Q_m(x,y)\cos(\frac{R_n(x,y)}{(x^2+y^2)^n}) + P_m(x,y)\sin(\frac{R_n(x,y)}{(x^2+y^2)^n}) \end{cases}$$

where P_m, Q_m and R_n are the homogeneous polynomials

$$\begin{array}{ll} P_m(x,y) = & Re((x+yi)^m) = x^m + \mathcal{O}(y^2), \\ Q_m(x,y) = & Im((x+yi)^m) = myx^{m-1} + \mathcal{O}(y^3), \\ R_n(x,y) = & Im((x-yi)^n) = -nyx^{n-1} + \mathcal{O}(y^3). \end{array}$$

If we do the following weighted blow-up $\{x = x, y = x^{n+1}v\}$ (see again [5]), we obtain

$$\begin{cases} \dot{x} = & P_m(x,x^{n+1}v)\cos(\eta) - Q_m(x,x^{n+1}v)\sin(\eta) \\ \dot{v} = & (Q_m(x,x^{n+1}v)\cos(\eta) + P_m(x,x^{n+1}v)\sin(\eta) \\ & -(n+1)x^n v\{P_m(x,x^{n+1}v)\cos(\eta) + Q_m(x,x^{n+1}v)\sin(\eta)\})/x^{n+1} \end{cases}$$

where $\eta(x,v) = R_n(x,x^{n+1}v)/(x^2 + x^{2(n+1)}v^2)^n$.

Clearly, $P_m(x,x^{n+1}v) = x^m P_m(1,x^n v)$, $Q_m(x,x^{n+1}v) = x^m Q_m(1,x^n v)$ and $R_n(x,x^{n+1}v) = x^n R_n(1,x^n v)$. Hence we have

$$\begin{cases} \dot{x} = & x^m(P_m(1,x^n v)\cos(\eta) + Q_m(1,x^n v)\sin(\eta)) \\ \dot{v} = & x^{m-(n+1)}(Q_m(1,x^n v)\cos(\eta) - P_m(1,x^n v)\sin(\eta) \\ & -(n+1)vx^{m-1}\{P_m(1,x^n v)\cos(\eta) - Q_m(1,x^n v)\sin(\eta)\}) \end{cases}$$

and dividing by $x^{m-(n+1)}$, we obtain

$$\begin{cases} \dot{x} = & x^{n+1}(P_m(1,x^n v)\cos(\eta) - Q_m(1,x^n v)\sin(\eta)) \\ \dot{v} = & Q_m(1,x^n v)\cos(\eta) + P_m(1,x^n v)\sin(\eta) \\ & -(n+1)x^{m-1}v\{P_m(1,x^n v)\cos(\eta) - Q_m(1,x^n v)\sin(\eta)\} \end{cases} \tag{2.14}$$

Some computations show that $P_m(1,x^n v) = 1 + \mathcal{O}(x^{2n}v^2)$, $Q_m(1,x^n v) = nx^n v + \mathcal{O}(x^{3n}v^3)$ and

$$\eta(x,v) = \frac{x^n R_n(1,x^n v)}{x^{2n}(1+x^{2n}v^2)^n} = \frac{x^n(-nx^n v + \mathcal{O}(x^{3n}v^3))}{x^{2n}(1+x^{2n}v^2)^n} = \frac{v(-n+\mathcal{O}(x^{2n}v^2))}{(1+x^{2n}v^2)^n}.$$

Using the above expressions, we have $P_m(1,0) = 1$, $Q_m(1,0) = 0$, and $\eta(0,v) = -nv$. Thus, there are infinitely many singular points on $x = 0$ located at $(0, k\pi/n)$, where $k \in \mathbb{Z}$. Their linear parts are given by

$$\begin{pmatrix} 0 & 0 \\ \star & -n(-1)^k \end{pmatrix},$$

so they are not hyperbolic. To obtain their local phase portrait, we first do the change of variables $\{x = x,\ u = v - k\pi/n\}$, $k \in \mathbb{Z}$ to translate the singular point

$(0, k\pi/n)$ to the origin of (x, u), and second we scale the time $\tau = (-n)t$. So, system (2.14) becomes

$$\frac{dx}{d\tau} = \frac{(-1)^{k+1}}{n} x^{n+1}(1 + \mathcal{O}(x)),$$
$$\frac{du}{d\tau} = u(1 + \mathcal{O}(x, u)).$$

If we apply the classification theorem of singular points with a zero eigenvalue (see Theorem 1 in page 149 in [17]), we see that if $n+1$ is even the local phase portrait at $(0, k\pi/n)$, where $k \in \mathbb{Z}$, is a saddle node, while if $n+1$ is odd the local phase portrait at $(0, k\pi/n)$, $k \in \mathbb{Z}$ is a node or a saddle depending on k odd or k even, respectively. Of course, going back through the directional blow up we obtain different local phase portraits at $z = 0$ depending on the parity of n, k and m. As expected, the origin admits infinitely many characteristic directions with an orbit tending to the origin in forward or backward time (determined by the infinitely many singular points after the blow up).

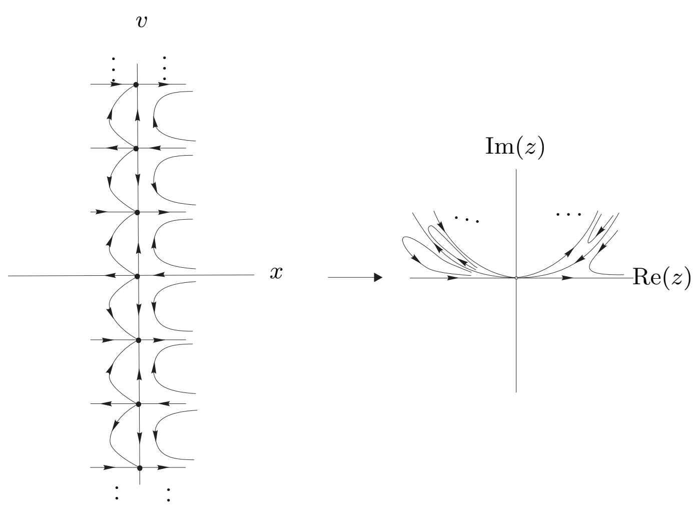

FIGURE 3. The phase portrait at the origin of equation (2.13) when $n = 1$ and $m = 2$.

To illustrate not only the local behavior around the origin but the phase portrait in the Poincaré disc, we consider the particular case $\dot{z} = z^2 \exp(1/z)$. On one hand, the above discussion shows that the local phase portrait around the origin is given by Figure 3. Moreover, to study the dynamics around infinity (see Section 2.3), we do the change of variables $z = 1/w$. Easily we get $\dot{w} = -e^w$ and Theorem 1.1 implies that the dynamics around $w = 0$ is governed by $\dot{w} = -1$. So, the dynamics around infinity is governed by $\dot{z} = z^2$. On the other hand, the equation is integrable and the general solution of the Chauchy problem $z(0) = z_0$ is

$$z(t) = \frac{-1}{\log(t + e^{-\frac{1}{z_0}})},$$

where log denotes the principle determination of the complex logarithm. It is easy to see that any initial condition (not lying in the invariant straight line $\mathrm{Im}(z) = 0$)

gives an orbit landing at $z = 0$ in positive, as well as, negative time. Notice that no solution is defined for all negative time (and all are defined for all positive time).

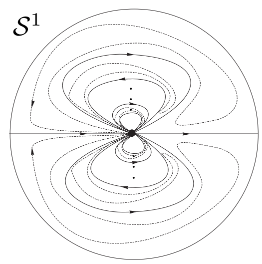

FIGURE 4. The global phase portrait in the Poincaré disc of equation (2.13) with $n = 1$ and $m = 2$.

By adding all together, we obtain in Figure (4) the scheme of the global phase portrait of equation $\dot{z} = z^2 \exp(1/z)$ in the Poincaré sphere.

3. **Limit cycles and polycycles.** The previous sections studied the dynamics associated with equation (1.1) near a singular point. We now turn our attention to the dynamics near either a periodic orbit, or a *graph*. We say that Γ is a graph of equation (1.1) if it is a compact connected invariant set such that the α and ω–limit set of every point in the regular orbits of Γ is a singular point, a pole or an essential singularity of Γ. Note that the flow is not defined on the singularities associated with Γ. We say that Γ is a polycycle if it is a *monodromic graph*, i.e. a graph for which the Poincaré map is well defined on at least one of its half-neighborhoods[1].

Indeed, it is known (see [10, 20, 6]) that for an analytic or a meromorphic function f, equation (1.1) has no limit cycles. Moreover, all periodic orbits fulfilling a plane region have the same period. This of course fully characterizes the local phase portrait near a periodic orbit. The more usual proof for this fact is based on the following argument. Let C be a periodic orbit of period τ. We take an appropriate tubular neighborhood U of C in which the function f is holomorphic (there are no poles or essential singularities). The solutions of the differential equation as well as the Poincaré map P and the τ-time function T are therefore also analytic functions in U. Let $z_1 \in C$ and fix V such that $z_1 \in V \subset U$. Since $T(z) = z$ for all points in $V \cap C$, then $T(z) \equiv z$ in V. Consequently, $P(z) \equiv z$. Therefore, there are no limit cycles and all periodic orbits near C have the same period.

Our approach shows that equation (1.1) is indeed a Hamiltonian differential equation with singularities given by the zeros and essential singularities of f. In particular, since any periodic orbit is isolated from such singularities the Hamiltonian structure implies the non-existence of limit cycles. Moreover, by playing with the Hamiltonian associated with f, we will be able to extend our study near periodic orbits to a study near monodromic graphs (or polycycles).

Theorem 1.3 follows from the following two propositions.

[1] A half-neighborhood means a ring-shape open set with Γ or some subset of Γ as one of its boundaries. For instance, if Γ has a number eight shape, then we can construct 3 such half-neighborhoods.

Proposition 3.1. *Consider the differential equation $\dot{z} = f(z)$. The following holds:*

(a) The above differential equation and $\dot{z} = if(z)$ commute, i.e. $[f, if] = 0$, whenever the function f is defined.

(b) Let C be a periodic orbit of period T. All orbits in a neighborhood of C are also periodic and have the same period T. Moreover,

$$T = \int_C \frac{1}{f(z)}\, dz.$$

(c) The equation $\dot{z} = f(z)$ has $(f\bar{f})^{-1}$ as an integrating factor. That is, its phase portrait is topologically equivalent to the one of equation $\dot{z} = 1/\overline{f(z)}$ whenever the integrating factor is defined (outside zeros and essential singularities of f).

Proof. (a) If we consider the differential equations $\dot{z} = f(z, \bar{z})$, $\dot{z} = g(z, \bar{z})$, where f and g are two analytic maps (not necessarily holomorphic), then the Lie bracket operator (2.4) becomes in this notation

$$[f, g] = fg_z - f_z g + \bar{f} g_{\bar{z}} - \bar{g} f_{\bar{z}}, \tag{3.15}$$

where the subscripts denote the corresponding derivatives of f and g with respect to z and $\bar{z}$. Easily, if f and g are holomorphic, (3.15) simplifies to

$$[f, g] = fg_z - f_z g. \tag{3.16}$$

Thus it is straightforward that $[f, if] = 0$ whenever the function f is defined.

Proof of (b). The equality given above $[f, if] = 0$ was already observed in [18] and used to prove isochronicity of the centers of holomorphic equations. This idea has been extended in [19, 21] to more general planar systems. In particular, it holds that if X has a non-degenerated center, Y is transversal to X in a punctured neighborhood of the critical point, and $[X, Y] = 0$ then the center is isochronous. Using the same idea as in those papers, namely the fact that the flows of $\dot{z} = f(z)$ and $\dot{z} = if(z)$ commute, we easily find that all orbits in a suitable neighborhood U of C are also periodic orbits, and all of them have the same period T. Let us compute this. Consider any periodic orbit $\gamma \in U$. Its period T_γ is given by

$$\int_\gamma \frac{1}{f(z)}\, dz = \int_0^{T_\gamma} \frac{z'(t)}{f(z(t))}\, dt = \int_0^{T_\gamma} dt = T_\gamma.$$

Notice that, using the Residues Theorem, the left hand side of this equation is determined by the sum of the residues of $1/f(z)$ over of the zeroes of f surrounded by γ, which of course is independent of γ. Hence, the above equality also gives that $T_\gamma = T$, *i.e.* another proof of the isochronism of all the periodic orbits in U.

Proof of (c). It is a straightforward computation that

$$\dot{z} = \frac{f(z)}{f(z)\overline{f(z)}} = \frac{1}{\overline{f(z)}}$$

is Hamiltonian wherever it is defined. Although the proof is ended, we would like explain how we know that equation (1.1) has $(f\bar{f})^{-1}$ as an integrating factor. We have used the following result, see [22]: Let X and Y be two differentiable vector fields in $U \in \mathbb{R}^2$ such that $v(x, y) = \det(X, Y)$ has no zeroes in U. Then, $[X, Y] = \lambda X$ if and only if v^{-1} is an integrating factor of X. Applying this result to our case ($\lambda = 0$), we obtain that the inverse of $\det(f, if) = f\bar{f}$ is an integrating factor for equation $\dot{z} = f(z)$. ☐

From Theorem 3.1.(c) we know that equation (1.1) admits an integrating factor outside the zeros or essential singularities of f, where equation (1.1) may have singular points whose local phase portrait does not preserve the area. Since such points are "far" from periodic orbits, we easily observe the non-existence of isolated periodic orbits. We would like to extend this result to polycycles. If a graph has a zero of f, it cannot be monodromic since the local phase portrait of a zero of f has no hyperbolic sectors. Thus, the singularities on the polycycle are either poles or essential singularities. If all were poles the integrating factor would be well defined and we could argue, as in the case of the periodic orbit, that, in the set in which the return map is defined, all the orbits should be periodic. The study of the case of polycycles with some essential singularity on them is more delicate. We prove:

Proposition 3.2. *Let Γ be a polycycle of equation (1.1). Then, in any half-neighborhood of Γ where the return map is defined, all orbits must be periodic.*

Proof. The results of Corollary 2.7 imply that the singularities over Γ cannot be zeroes of f. Thus, all singular points in Γ are either poles or essential singularities of of f.

Take a small open half-neighborhood U of Γ in which the return map is defined and there are neither zeroes, nor poles, nor essential singularities of f. On U, instead of considering $\dot{z} = f(z)$, we use Theorem 3.1.(c) and we can study the solutions of the Hamiltonian system $\dot{z} = 1/\overline{f(z)}$.

From Theorem 3.1.(b) we know that if C is a periodic orbit, all orbits in a small enough neighborhood of it are also periodic. Thus, in U either all orbits are periodic or the polycycle (or one of its sub-polycycles) is attracting or repelling. In other words, the case of infinitely many periodic orbits accumulating to some part of Γ can be avoided unless all the solutions in U are periodic. Thus, either U is full of periodic orbits or, without loss of generality, all the solutions starting at U tend to Γ. Hence we can assume that U is positively invariant.

First we prove that all the solutions of $\dot{z} = 1/\overline{f(z)}$ starting at U are defined for all positive time. Denote by $\varphi(q;t)$ the solution starting at q for time t, and by $\gamma_q^+ = \{\varphi(q;t) : t \in [0,\infty\}$. Take any piece of Γ made up only of regular points of f, with extrema g_1 and g_2, at which γ_q^+ accumulates. Let $\tau > 0$ be the time spent for the flow of the differential equation to pass from g_1 to g_2 through Γ. By continuous dependence of the flow with respect to initial conditions, there is a neighborhood W of g_1 such that all solutions starting at W are defined for $t > \tau/2$. Thus, since any trajectory $\varphi(q;t)$ starting at $q \in U$ passes infinitely many times through W, we obtain that the positive maximal interval of definition of $\varphi(q;t)$ is $[0,\infty)$.

Take any closed ball B_0 of area k totally contained in U, and $n \in \mathbb{N}$ such that $nk > \text{area}(U)$. Since the flow of $\dot{z} = 1/\overline{f(z)}$ preserves area, when we consider the n sets $\{\varphi(V_0, iT)\}_{i=0,1,\ldots,n-1}$, there always exist at least two of them that intersect. Thus, for some $0 \le k < \ell \le n-1$, $\varphi(B_0;kT) \cap \varphi(B_0;\ell T) \neq \emptyset$. Since the flow is a diffeomorphism $B_1 := B_0 \cap \varphi(B_0;m_0T) \neq \emptyset$ where $m_0 = \ell - k$. This B_1 is a compact set included in B_0. Repeating this procedure infinitely many times we get a non-empty compact set $B := \cap_{i=0}^{\infty} B_i$. If $x \in B \subset B_0$ there exists an unbounded sequence of positive integers $\{k_n\}_{n\in\mathbb{N}}$ such that $\varphi(x, k_nT) \in B_0$ for all $n \in \mathbb{N}$. Since the flow is defined in a planar ring-shape domain, this property implies that all orbits cutting B are periodic orbits. Hence, since U contains some periodic orbits, it is full of periodic orbits. Notice that in the above reasoning is essentially the same as in the proof of Poincaré recurrence Theorem (see, for instance, [8, Thm. 11.4]). ☐

Remark 3.3. We notice that, in [14], the authors provide examples of planar systems which are Hamiltonian in the complement of finitely many algebraic curves, where they present limit cycles. This is totally coherent with our proof of the above theorem because, in their examples the authors show that the trajectories tending to these stable (unstable) limit cycles are not defined for all positive (negative) time.

4. **Bifurcation diagram of a rational family.** In this section we give a topological classification of the phase portrait of the family of rational functions of the form

$$\dot{z} = \frac{P(z)}{Q(z)} \ , \tag{4.17}$$

where P and Q are polynomials in the z variable such that $\deg(P) \leq 2$ and $\deg(Q) \leq 1$ without common factors. We divide the general result into two propositions depending on the degree of the denominator of system (4.17). First we consider the case of the polynomial Q of degree 0, so that it corresponds to the polynomial case (see [6]). In the following lemma we show a normal form of these equations.

Lemma 4.1. *Let $P(z)$ and $Q(z)$ be two polynomials such that $\deg(P) \leq 2$ and $Q(z) = d$, (i.e., $\deg(Q) = 0$), then the corresponding equation (4.17) is conformally conjugated to*
(a) $\dot{z} = 1$ if $P(z) = c$,
(b) $\dot{z} = kz$ if $P(z) = c(z - c_0)$, where $k = c/d \in \mathbb{C}$,
(c) $\dot{z} = z(z - \xi)$ if $P(z) = c(z - c_0)(z - c_1)$, where $\xi = c(c_1 - c_0)/d \in \mathbb{C}$.

The following proposition gives the phase portrait of the normal forms given in Lemma 4.1.

Proposition 4.2. *The following conditions hold:*

(a) The phase portrait of equation $\dot{z} = 1$ in the Poincaré disc is topologically equivalent to Figure 5(1).
(b) Let $k \in \mathbb{C}$. The phase portrait of equation $\dot{z} = kz$ in the Poincaré disc is topologically equivalent to
  (b1) Figure 5(2) if $\mathrm{Re}(k) = 0$,
  (b2) Figure 5(3) if $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$,
  (b3) Figure 5(4) if $\mathrm{Im}(k) = 0$.
(c) Let $\xi \in \mathbb{C}$. The phase portrait of equation $\dot{z} = z(z - \xi)$ in the Poincaré disc is topologically equivalent to
  (c1) Figure 5(5) if $\xi \neq 0$, and $\mathrm{Re}(\xi) = 0$,
  (c2) Figure 5(6) if $\xi \neq 0$, and $\mathrm{Re}(\xi) \neq 0$,
  (c3) Figure 5(7) if $\xi = 0$.

Proof. Statement (a) is straightforward.

First we study the point of infinity. From Corollary 2.9, the phase portrait of equation (4.17) in a neighborhood of infinity corresponds to $\dot{z} = kz$. Consequently, the phase portrait in the Poincaré disc is given by Figure 5(2) if $\mathrm{Re}(k) = 0$, by Figure 5(3) if $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$, or by Figure 5(4) if $\mathrm{Im}(k) = 0$.

From Corollary 2.9, the phase portrait of equation (4.17) in a neighborhood of infinity corresponds to $\dot{z} = z^2$. Moreover, there are either two singular points at $z = 0$ and $z = \xi$ (case $\xi \neq 0$), or just one singular point at $z = 0$ (case $\xi = 0$). In the first case, the derivative at $z = 0$ and $z = \xi$ is $L_0 = -\xi$ and $L_\xi = \xi$, respectively. Hence, from Theorem 2.6(a), the phase portrait in the Poincaré sphere is topologically equivalent to either Figure 5(5) or (6) depending on whether $\mathrm{Re}(L_\xi) = \mathrm{Re}(\xi) = 0$, or $\mathrm{Re}(L_\xi) = \mathrm{Re}(\xi) \neq 0$, respectively. If $\xi = 0$, from Theorem 2.6(b), the phase portrait in the Poincaré disc is topologically equivalent to Figure 5(7). □

Next we consider the case in which $Q(z)$ is a polynomial of degree 1. In the following lemma we obtain a normal form for these equations.

Lemma 4.3. *Let $P(z)$ and $Q(z)$ be two polynomials such that $deg(P) \leq 2$ and $Q(z) = d(z - d_0)$ (i.e., $deg(Q) = 1$), then the corresponding equation (4.17) is conformally conjugated to*

(a) $\dot{z} = 1/z$ *if* $P(z) = c$
(b) $\dot{z} = (z - \xi)/z$ *if* $P(z) = c(z - c_0)$, *where* $\xi = d(c_0 - d_0)/c$,
(c) $\dot{z} = k(z - \xi)(z - 1)/z$ *if* $P(z) = c(z - c_0)(z - c_1)$, *where* $k = c/d$ *and* $\xi = (c_1 - d_0)(c_0 - d_0)$.

The following proposition provides the phase portrait of the normal forms given in Lemma 4.3.

Proposition 4.4. *The following conditions hold*

- *(a) The phase portrait of equation $\dot{z} = 1/z$ in the Poincaré disc is topologically equivalent to Figure 5(8).*
- *(b) The phase portrait of equation $\dot{z} = (z - \xi)/z$ in the Poincaré disc is topologically equivalent to*
  - *(b1) Figure 5(9) if* $\mathrm{Re}(L_\xi) = 0$,
  - *(b2) Figure 5(10) if* $\mathrm{Re}(L_\xi) \neq 0$,

  where $L_\xi = 1/\xi$ is the linear part at the singular point $z = \xi$.
- *(c) The phase portrait of equation $\dot{z} = k(z - 1)(z - \xi)/z$ in the Poincaré disc is topologically equivalent to*
  - *(c1) Figure 5(11) if* $\mathrm{Re}(k) = 0$, $\mathrm{Re}(L_1) = \mathrm{Re}(L_\xi) = 0$ *and* $Re(\xi) < 0$,
  - *(c2) Figure 5(12) if* $\mathrm{Re}(k) = 0$, $\mathrm{Re}(L_1) = \mathrm{Re}(L_\xi) = 0$ *and* $Re(\xi) > 0$,
  - *(c3) Figure 5(13) if* $\mathrm{Re}(\mathrm{k}) = 0$ *and* $\mathrm{Re}(L_1)\mathrm{Re}(L_\xi) \neq 0$,
  - *(c4) Figure 5(14) if* $\mathrm{Re}(\mathrm{k}) = 0$ *and* $\xi = 1$,
  - *(c5) Figure 5(15) if* $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) = 0$,
  - *(c6) Figure 5(16) if* $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) < 0$,
  - *(c7) Figure 5(17) if* $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) > 0$,
  - *(c8) Figure 5(18) if* $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$ *and* $\xi = 1$,
  - *(c9) Figure 5(19) if* $\mathrm{Im}(k) = 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) = 0$,
  - *(c10) Figure 5(20) if* $\mathrm{Im}(k) = 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) < 0$,
  - *(c11) Figure 5(21) if* $\mathrm{Im}(k) = 0$ *and* $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) > 0$,
  - *(c12) Figure 5(22) if* $\mathrm{Im}(k) = 0$ *and* $\xi = 1$,

  where $L_1 = k(1 - \xi)$ and $L_\xi = k(\xi - 1)/\xi$ are the linear part at the singular points $z = 1$ and $z = \xi$, respectively.

Proof. Since the proof is quite long we divide it into three cases.

Proof of (a) In this case, there are no finite singular points and the origin is a pole of order one which, from Theorem 1.1, has local phase portrait that is topologically equivalent to a hyperbolic saddle. On the other hand, from Corollary 2.9, the phase

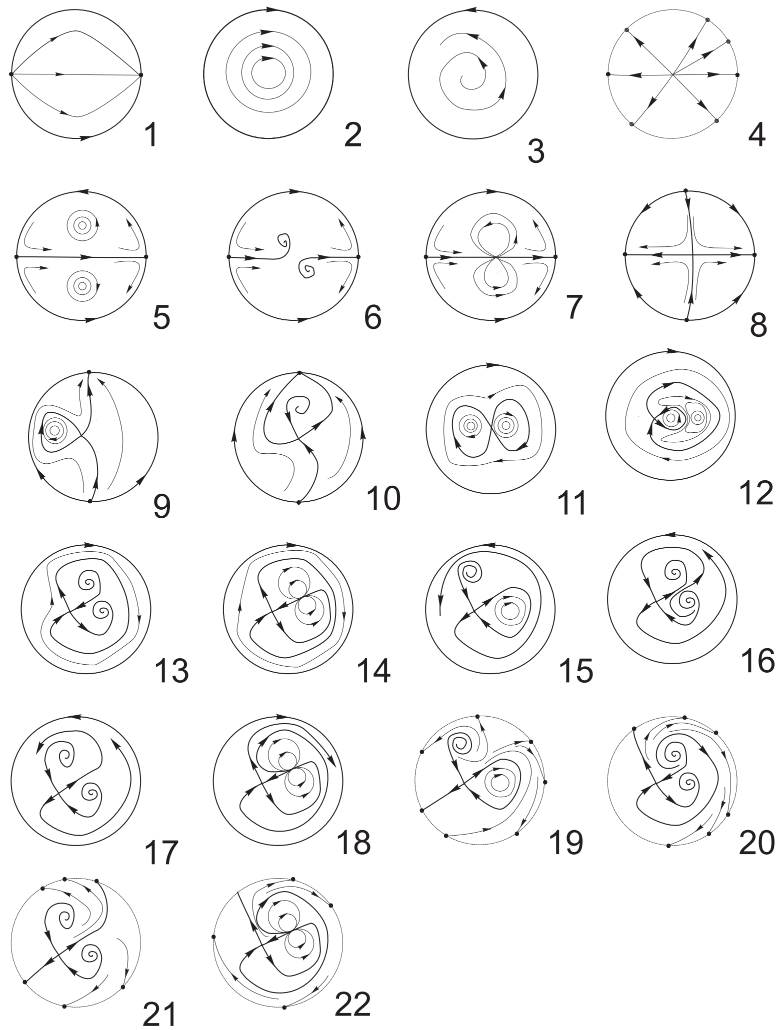

FIGURE 5. Topological classification of the rational equation (4.17).

portrait of equation (4.17) in a neighborhood of infinity corresponds to $\dot{z} = 1/z$. Easily, the topological phase portrait in the Poincaré disc is given by Figure 5(8).

Proof of (b) We have that the origin is a pole of order one and the local phase portrait is topologically equivalent to a hyperbolic saddle. The point $z = \xi$ is a singular point with a linear part given by $L_\xi = 1/\xi$. Hence, it is a center if $\mathrm{Re}(L_\xi) = 0$, or it is a stable or unstable focus if $\mathrm{Re}(L_\xi) \neq 0$. Moreover, from Corollary 2.9, the phase portrait of equation (4.17) in a neighborhood of infinity corresponds to $\dot{z} = 1 + \xi/z$ at infinity. Taking all of this into account, we can easily check that the only topological phase portraits in the Poincaré disc are either Figure 5(9) if $\mathrm{Re}(L_\xi) = 0$, or Figure 5(10) if $\mathrm{Re}(L_\xi) \neq 0$.

Proof of (c) In this case, the origin is a pole of order one and the local phase portrait is topologically equivalent to a hyperbolic saddle. If $\xi \neq 1$, then $z = 1$ and $z = \xi$ are singular points with a linear part given by $L_1 = k(1 - \xi)$ and $L_\xi = k(\xi - 1)/\xi$.

Moreover, we have that $L_1 + \xi L_\xi = 0$. If $\xi = 1$, there is a unique singular point and the local phase portrait is given by the union of two elliptic sectors (see Theorem 2.6). From Corollary 2.9 the phase portrait of equation (4.17) in a neighborhood of infinity corresponds to $\dot{z} = kz$. The behavior of equation $\dot{z} = kz$ depends on $\mathrm{Re}(k) = 0$, $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$, or $\mathrm{Im}(k) = 0$, respectively. We study such cases independently.

In what follows $\xi = \xi^R + i\xi^I$, $r = |\xi|^2 = \xi\bar{\xi}$ and $k = \alpha + i\beta$.

(i) Case $\mathrm{Re}(k) = 0$, i.e. infinity is a *periodic orbit.* It is easy to see that

$$\mathrm{Re}\,(L_1) = \beta\xi^I$$
$$\mathrm{Re}\,(L_\xi) = -\beta\xi^I/r.$$

If $\mathrm{Re}(L_1) = \mathrm{Re}(L_\xi) = 0$, then the two centers are located on the real line. The phase portrait of equation(4.17) in the Poincaré disc is therefore topologically equivalent to either Figure 5(11) or Figure 5(12) depending on whether $\xi < 0$ or $\xi > 0$.

If $\mathrm{Re}(L_1)\mathrm{Re}(L_\xi) < 0$ we have one stable and one unstable singular point. The phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(13). Notice that, from Theorem 1.3, when we have one stable and one unstable singular point (and the pole at the origin) there are no other topological configurations since no polycycle can be asymptotically stable or unstable.

If, finally, $\xi = 1$, the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(14).

(ii) Case $\mathrm{Re}(k)\mathrm{Im}(k) \neq 0$ i.e. infinity is a stable or unstable *limit cycle at infinity.* Some computations show that

$$\mathrm{Re}\,(L_1) = \alpha(1 - \xi^R) + \beta\xi^I \quad \text{and}$$
$$\mathrm{Re}\,(L_\xi) = (\alpha(\xi^R(\xi^R - 1) + (\xi^I)^2) - \beta\xi^I)/r.$$

The only solution of the equation $\mathrm{Re}\,(L_1) = \mathrm{Re}\,(L_\xi) = 0$ is $\xi^R = 1$ and $\xi^I = 0$. So two centers are impossible.

Let $\xi \neq 1$. If $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) = 0$ (one center and one stable/unstable singular point), the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(15). If $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) < 0$ (one stable and one unstable singular point), the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(16). If $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) > 0$ (two stable or two unstable singular points), the phase portrait is topologically equivalent to Figure 5(17).

Let $\xi = 1$. Easily, the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(18). Proposition 3.2 clearly shows that it is not possible to have a separatrix connection because there would be a stable or unstable polycycle.

(iii) Case $\mathrm{Im}(k) = 0$ i.e. infinity is *degenerate* (filled of singular points). Some computations show that

$$\mathrm{Re}\,(L_1) = \alpha(1 - \xi^R) \quad \text{and}$$
$$\mathrm{Re}\,(L_\xi) = \alpha(\xi^R(\xi^R - 1) + (\xi^I)^2)/r.$$

As before, it is not possible to have two centers.

Let $\xi \neq 0$. If $\mathrm{Re}\,(L_1)\,\mathrm{Re}\,(L_\xi) = 0$ (one center and one stable or unstable focus), the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(19). If $\mathrm{Re}\,(L_\xi)\,\mathrm{Re}\,(L_1) < 0$ or $\mathrm{Re}\,(L_\xi)\,\mathrm{Re}\,(L_1) > 0$ the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(20) or (21), respectively.

Let $\xi_0 = 1$. It is straightforward to see that the phase portrait of equation (4.17) in the Poincaré disc is topologically equivalent to Figure 5(22). □

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