# Weak commutativity and the group $\nu(G)$

## around Rocco's constructions

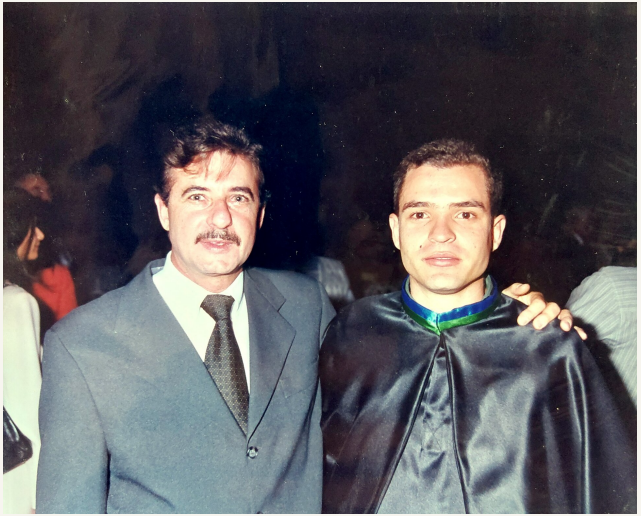
Ricardo Nunes de Oliveira

Universidade Federal de Goiás

Group Theory and Applications — Noraí R. Rocco 75th Birthday Conference

*Dedicated to Professor Noraí Romeu Rocco on his 75th birthday*

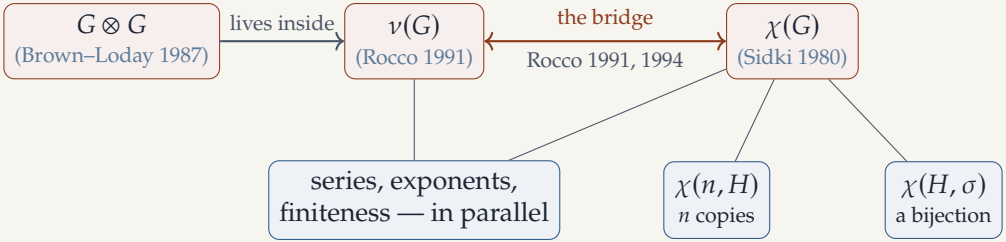
# Where it started: my graduation



My undergraduate graduation, with Professor Noraí

# The constructions in this talk

**I.** Sidki and Rocco




**II.** joint work

# My first GAP routine

Master's thesis (UnB), advisor Noraí R. Rocco: *Central extensions of groups and the Peiffer product*.

```
LoadPackage("XMod");;
a:=(1,2,3,4); b:=(2,4);
x:=(1,2,4,7)(3,6,8,5);
y:=(1,3,4,8)(2,5,7,6);
r:=(1,3)(2,4); s:=(1,2)(3,4);
d8:=Group(a,b); q8:=Group(x,y);
v4:=Group(r,s);
autq8:=AutomorphismGroup(q8);
qd:=GroupHomomorphismByImages(
  d8, autq8, [a,b],
  [InnerAutomorphism(q8,x),
   InnerAutomorphism(q8,y)]);
d8q8:=SemidirectProduct(d8,qd,q8);
I:=Embedding(d8q8,1);
J:=Embedding(d8q8,2);
autd8q8:=AutomorphismGroup(d8q8);
```

```
f1:=GroupHomomorphismByImages(
  d8q8, d8q8, [a^I,b^I,x^J,y^J],
  [a^I,(a^2*b)^I,x^J,(x^2*y)^J]);
f2:=GroupHomomorphismByImages(
  d8q8, d8q8, [a^I,b^I,x^J,y^J],
  [(a^I)^3,b^I,(x^J)^3,y^J]);
partial1:=GroupHomomorphismByImages(
  d8q8, v4, [a^I,b^I,x^J,y^J], [r,s,r,s]);
d8q8v:=GroupHomomorphismByImages(
  v4, autd8q8, [r,s], [f1,f2]);
X0:=PreXModByBoundaryAndAction(
  partial1, d8q8v);
P:=PeifferSubgroup(X0);
X1:=XModByPeifferQuotient(X0);
pr:=NaturalHomomorphismByNormalSubgroup(
  d8q8, P);
i:=CompositionMapping2(pr,I);
j:=CompositionMapping2(pr,J);
d8pfq8:=Image(pr);
```

**Whitehead:** do $G$, $H$ embed in $G \bowtie H$?

$D_8 \hookrightarrow D_8 \bowtie Q_8 \hookleftarrow Q_8$

# GAP: do $D_8$ and $Q_8$ embed in $D_8 \bowtie Q_8$?

```
gap> LoadPackage("XMod");;
gap> Read("d8pfq8");
gap> IsInjective(i);IsInjective(j);
true
true
```

Both embeddings are injective.

# GAP: the pre-crossed module $X_0$

```
gap> Display(X0);

Pre-crossed module :-
: Source group has generators:
  [ ( 3, 8)( 5, 6)( 9,10,11,12), ( 2, 7)( 5, 6)(10,12), (1,2,4,7)(3,6,8,5),
  (1,3,4,8)(2,5,7,6) ]
: Range group has generators:
  [ (1,3)(2,4), (1,2)(3,4) ]
: Boundary homomorphism maps source generators to:
  [ (1,3)(2,4), (1,2)(3,4), (1,3)(2,4), (1,2)(3,4) ]
: Action homomorphism maps range generators to automorphisms:
  (1,3)(2,4) --> { source gens -->
[ ( 3, 8)( 5, 6)( 9,10,11,12), ( 2, 7)( 5, 6)( 9,11), (1,2,4,7)(3,6,8,5),
  (1,8,4,3)(2,6,7,5) ] }
  (1,2)(3,4) --> { source gens -->
[ ( 3, 8)( 5, 6)( 9,12,11,10), ( 2, 7)( 5, 6)(10,12), (1,7,4,2)(3,5,8,6),
  (1,3,4,8)(2,5,7,6) ] }
  These 2 automorphisms generate the group of automorphisms.
```

# GAP: the crossed module $X_1$ (Peiffer quotient)

```
gap> Display(X1);

Crossed module :-
: Source group has generators:
  [ f1, f2, f3, f4 ]
: Range group has generators:
  [ (1,3)(2,4), (1,2)(3,4) ]
: Boundary homomorphism maps source generators to:
  [ (1,2)(3,4), (1,3)(2,4), (1,2)(3,4), (1,3)(2,4) ]
: Action homomorphism maps range generators to automorphisms:
  (1,3)(2,4) --> { source gens --> [ f1*f5, f2, f3*f5, f4 ] }
  (1,2)(3,4) --> { source gens --> [ f1, f2*f5, f3, f4*f5 ] }
  These 2 automorphisms generate the group of automorphisms.
```

# GAP: the Peiffer product $D_8 \bowtie Q_8$

```
gap> IdGroup(d8pfq8);StructureDescription(d8pfq8);
[ 32, 48 ]
"C2 x ((C4 x C2) : C2)"
```

$$D_8 \hookrightarrow D_8 \bowtie Q_8 \hookleftarrow Q_8$$

$|D_8 \bowtie Q_8| = 32$ (`SmallGroup(32,48)`)

# The group $\nu(G)$

Rocco 1991

Let $G^\varphi$ be a copy of $G$ via $\varphi$. Define

$$\nu(G) = \left\langle G \cup G^\varphi \;\middle|\; [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi},\ g_i \in G \right\rangle.$$

**Proposition** (Rocco, *Bol. Soc. Bras. Mat.* 22, 1991, Prop. 2.6)

$$[G, G^\varphi] \cong G \otimes G \quad \text{via} \quad [g, h^\varphi] \mapsto g \otimes h, \qquad \nu(G) = \big([G, G^\varphi] \cdot G\big) \cdot G^\varphi.$$

- $\Delta(G) = \langle [g, g^\varphi] \mid g \in G \rangle \leq Z(\nu(G)) \cap \nu(G)'$.
- $\Theta(G) = \ker\big(\nu(G) \to G,\ g, g^\varphi \mapsto g\big)$.

# The non-abelian tensor square, inside $\nu(G)$

Brown–Loday 1987, read in Rocco's $\nu(G)$

| **in** $G \otimes G$ | | **in** $\nu(G)$ |
|---|---|---|
| $g \otimes h$ | $\longleftrightarrow$ | $[g, h^\varphi]$ |
| $gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h)$ | $\longleftrightarrow$ | $[gg_1, h^\varphi] = [g, h^\varphi]^{g_1}\,[g_1, h^\varphi]$ |
| $g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$ | $\longleftrightarrow$ | $[g, (hh_1)^\varphi] = [g, h_1^\varphi]\,[g, h^\varphi]^{h_1^\varphi}$ |
| $\kappa : g \otimes h \mapsto [g, h]$ | $\longleftrightarrow$ | $[g, h^\varphi] \mapsto [g, h]$ (from $\nu(G) \to G$) |
| $\mu(G) = \ker \kappa \cong \pi_3(SK(G, 1))$ | $\longleftrightarrow$ | $\mu(G) = [G, G^\varphi] \cap \Theta(G)$ |
| $G \wedge G = G \otimes G / \langle g \otimes g \rangle$ | $\longleftrightarrow$ | $[G, G^\varphi] / \Delta(G)$ |

$$1 \to M(G) \to G \wedge G \to G' \to 1$$

The defining relations of $G \otimes G$ become commutator identities: tensor computations are commutator calculus in $\nu(G)$.

# The weak commutativity group $\chi(G)$

Sidki 1980

$$\chi(G) = \left\langle G \cup G^{\psi} \;\middle|\; [g, g^{\psi}] = 1,\ \forall g \in G \right\rangle.$$

$$L(G) = \langle g^{-1} g^{\psi} \mid g \in G \rangle \qquad \text{kernel of } \chi(G) \to G$$
$$D(G) = [G, G^{\psi}] \qquad \text{kernel of } \chi(G) \to G \times G$$
$$W(G) = L(G) \cap D(G)$$
$$R(G) = [G, L(G), G^{\psi}]$$

---

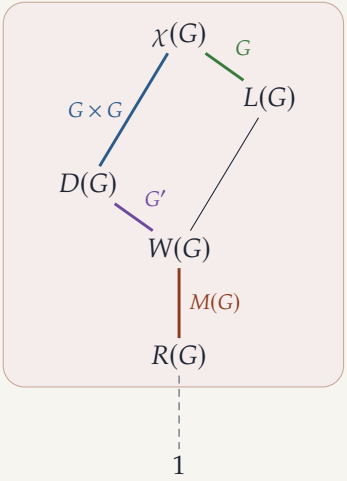
$[L(G), D(G)] = 1$ $\qquad$ $\chi(G)/W(G) \hookrightarrow G \times G \times G$

$R(G) \leq W(G)$ $\qquad$ $D(G)/R(G) \cong G \wedge G$

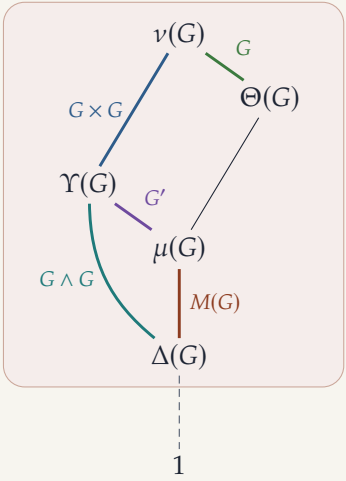
$W(G)/R(G) \cong M(G)$ (Rocco 1982, via Miller)

---

# The bridge




$$\frac{\chi(G)}{R(G)} \cong \frac{\nu(G)}{\Delta(G)}$$



# Two operators, in parallel

| if $G$ is ... | then $\chi(G)$ is ... (Sidki 1980) | then $\nu(G)$ is ... (Rocco 1991) |
|---|---|---|
| a finite $\pi$-group | a finite $\pi$-group | a finite $\pi$-group |
| solvable, length $\ell$ | solvable, length $\leq \ell + 1$ | solvable, length $\leq \ell + 1$ |
| nilpotent, class $c$ | nilpotent, if $G$ is finite | nilpotent, class $\leq c + 1$ |
| of order $p^n$ | $\lvert\chi(G)\rvert \mid p^{2n+n(n-1)/2}$, $p$ odd (Rocco 1982) | $\lvert\nu(G)\rvert \mid p^{n^2+2n-mn}$, $\lvert G'\rvert = p^m$ |
| polycyclic | *later* | finite solvable: explicit pcp (Rocco 1994) |

Finiteness of $\nu(G)$ comes from $\chi(G)$, through the bridge.

# $\chi(G)$: nilpotent groups and $p$-groups

**Theorem** (Sidki, *J. Algebra* 63, 1980)

$H$ finite nilpotent, with Sylow subgroups $P_1, \dots, P_k \implies \chi(H) \cong \chi(P_1) \times \cdots \times \chi(P_k)$ is finite nilpotent: it all reduces to $p$-groups.

**Theorem** (Rocco, *J. Algebra* 76, 1982)

$p$ odd, $H$ a finite $p$-group of order $p^n$ and class $c$:

$$|\chi(H)| \,\Big|\, p^{2n} \cdot p^{n(n-1)/2}, \qquad \text{class } \chi(H) \le 2c, \qquad W(H)/R(H) \cong M(H).$$

For $p = 2$ the growth is exponential: $|\chi(C_2^k)| = 2^{2^k - 1} \cdot 2^k$ (Sidki).

**Theorem** (Gupta, Rocco, Sidki, *Illinois J. Math.* 30, 1986)

$G$ nilpotent, $m$-generated ($m \ge 2$), of class $\le c$:

$$m \le c+2: \quad \gamma_{c+3}(\chi(G)) = 1,$$
$$m \ge c+3: \quad \gamma_{c+3}(\chi(G)) \text{ elementary abelian 2-group, rank} \le \sum_{k=c+3}^{m} \binom{m}{k}.$$

# Three subgroups, on each side

**In $\nu(G)$**

(with Bastos, Monetta, Rocco, *J. Algebra* 598, 2022)

$\Upsilon_1 = [G, G^\varphi]$, $\Upsilon_2 = [\Theta, G]$, $\Upsilon_3 = [\Theta, G^\varphi]$.

- $\Upsilon_1 \cong \Upsilon_2 \cong \Upsilon_3 \cong G \otimes G$.
- $[\Upsilon_2, G^\varphi] = 1 = [\Upsilon_3, G]$ (Rocco 1994).

**In $\chi(G)$**

(with Bastos, Ortega, *J. Group Theory* 29, 2026)

$D = [G, G^\varphi]$, $L_1 = [L, G]$, $L_2 = [L, G^\varphi]$.

- $D/R \cong L_1/R \cong L_2/R \cong G \wedge G$.
- $[L_1, L_2] \leq R$.

$\nu(G)/\Delta(G) \cong \chi(G)/R(G)$:

$$\begin{aligned} \Upsilon_1 &\leftrightarrow D \\ \Upsilon_2 &\leftrightarrow L_1 \\ \Upsilon_3 &\leftrightarrow L_2 \\ \Theta &\leftrightarrow L \end{aligned}$$

# Series, in parallel

## $\nu(G)$

(with Bastos, Monetta, Rocco, *J. Algebra* 598, 2022)

- $\nu(G)' = \Upsilon_1\Upsilon_2\Upsilon_3$, central product
- $\nu(G)^{(k+1)} = \Upsilon_1^{(k)}\Upsilon_2^{(k)}\Upsilon_3^{(k)}$, $k \geq 0$, central product
- $\gamma_k(\nu(G)) = [\Upsilon_{1,k-2}G]\,[\Upsilon_{2,k-2}G]\,[\Upsilon_{3,k-2}G^{\varphi}]$, $k \geq 2$.

## $\chi(G)$

(with Bastos, Ortega, *J. Group Theory* 29, 2026)

- $\chi(G)' = D \cdot L_1L_2$, central product
- $\chi(G)^{(k+1)} = D^{(k)}L_1^{(k)}L_2^{(k)}$, $k \geq 2$, central product
- $\gamma_k(\chi(G)) = [D,_{k-2}G]\,[L_{1,k-2}G]\,[L_{2,k-2}G^{\varphi}]$, $k \geq 2$.

$\nu(G)'$ holds three copies of $G \otimes G$; $\chi(G)'$ holds three copies of $D$: $D \cong L_1 \cong L_2$ (new)

# Properties preserved by $\chi$

**Theorem** (with Lima, *J. Group Theory* 19, 2016)

If $H$ is polycyclic (resp. polycyclic-by-finite), then so is $\chi(H)$.

Corollary (through the bridge): $H$ polycyclic-by-finite $\Rightarrow$ $\nu(H)$ and $H \otimes H$ polycyclic-by-finite.

Beyond polycyclic: local properties.

**Theorem** (with Bastos, Lima, Nakaoka, *Comm. Algebra* 52, 2024)

The set $\{[g, h^{\varphi}]\}$ is finite iff $D(G)$ is finite. If $\mathcal{P}$ is closed under subgroups and quotients and preserved by $\chi$, then $\chi$ preserves locally $\mathcal{P}$ groups.

# $n$ isomorphic copies

with Sidki, Rocco, submitted, 2026

The mother group

$$\chi_n(H) = \left\langle H, H^{\psi}, H^{\psi_2}, \ldots, H^{\psi_{n-1}} \;\middle|\; [h^{\psi_i}, h^{\psi_j}] = 1,\ h \in H,\ i \neq j \right\rangle \qquad (\psi_0{=}1,\ \psi_1{=}\psi)$$

$\chi_3(C_2 \times C_2)$ is infinite: weak commutativity alone does not give finiteness.

$$\chi(n, H) = \left\langle\, \chi_n(H) \;\middle|\; [D_{ij}, L_{kl}] = 1,\ \ i \neq j,\ k \neq l \,\right\rangle$$

$$D_{ij} = [H^{\psi_i}, H^{\psi_j}], \quad L_{kl} = \langle h^{-\psi_k} h^{\psi_l} \mid h \in H \rangle.$$

Studied since my doctoral thesis (UnB, 2007).

# Order and structure

**Theorem** (with Sidki, Rocco, submitted, 2026)

If $H$ is finite, then $|\chi(n,H)|$ divides $|\chi(H)|^{\binom{n}{2}}\,|H|^{\,n-2\binom{n}{2}}$.

The bound is attained for $H$ elementary abelian, $p$ odd.

**Theorem** (with Sidki, Rocco, submitted, 2026)

If $H$ is perfect, then $D_{ij} = D_{ks} \cong D(H)$ for all pairs, and

$$\chi(n,H) \cong \left(\cdots\left((D\cdot H)\cdot H^{\psi}\right)\cdots\right)\cdot H^{\psi_{n-1}} \cong T_n(\tilde{H}).$$

# Nilpotency of $\chi(n, H)$

**Theorem** (with Sidki, Rocco, submitted, 2026)

$H$ nilpotent, $m$-generated ($m \geq 2$), of class $\leq c$, and $G = \chi(n, H)$:

$$
\begin{aligned}
m = 2: &\quad \gamma_{c+2}(G) = 1, \\
2 < m \leq c+2: &\quad \gamma_{c+3}(G) = 1, \\
m \geq c+3: &\quad \gamma_{c+3}(G) \text{ elementary abelian 2-group, rank} \leq \sum_{k=c+3}^{m} \binom{m}{k}\binom{n}{2}.
\end{aligned}
$$

# Does $\chi(H)$ embed in $\chi(n, H)$?

with Sidki, Rocco, submitted, 2026

$$\chi(H) \twoheadrightarrow \langle H, H^{\psi} \rangle \leq \chi(n, H)$$

Is it injective?

- Yes when $H$ is abelian, perfect, or $R(H) = 1$.
- Not in general (GAP), already in $\chi(3, H)$:

| $H$ | $\lvert\langle H, H^{\psi}\rangle\rvert$ | $\lvert\chi(H)\rvert$ | kernel |
|---|---|---|---|
| $(C_4 \times C_4) \rtimes C_2$ `[32,34]` | $2^{17}$ | $2^{18}$ | $C_2$ |
| $C_3 \times E$ `[81,12]` | $3^{13}$ | $3^{14}$ | $C_3$ |

$E$: extraspecial of order 27 and exponent 3. The failure happens for odd primes too.

# Weak commutativity by a bijection

with Sidki 2009; work in progress

For a bijection $\sigma\colon H \setminus \{1\} \to \dot{H} \setminus \{1\}$,

$$\chi(H,\sigma) = \left\langle H,\ \dot{H} \;\middle|\; [h, h^\sigma] = 1,\ \forall h \in H \right\rangle.$$

$\chi(H,\sigma) \twoheadrightarrow H \times \dot{H}$ always, $\quad \chi(H,\sigma) = \chi(H)$ if $\sigma$ is an isomorphism.

**Conjecture** (with Sidki, *Bull. Braz. Math. Soc.* 40, 2009)

If $H$ is finite nilpotent, so is $\chi(H,\sigma)$; if $H$ is a finite $p$-group, so is $\chi(H,\sigma)$.

| if $H$ is ... | then $\chi(H,\sigma)$ ... |
|---|---|
| cyclic of order $n$ | $\cong C_n \times C_n$, for *every* $\sigma$ |
| finite | $\chi(H,\sigma)/\chi(H,\sigma)''$ is nilpotent of class $\leq \lvert H\rvert$ |
| $C_p^n$, $p$ odd, $n \geq 3$, $\sigma$ a transposition | is a quotient of $\chi(H)$ |

# Between $H \times \dot{H}$ and $\chi(H)$

Up to equivalence, fewer bijections to test:

| | |
|---|---|
| any $H$ | $\chi(H, \alpha\sigma\beta) \cong \chi(H, \sigma)$ for $\alpha, \beta \in \mathrm{Aut}(H)$: one $\sigma$ per double coset |
| $H$ elementary abelian | a basis can be fixed: $a_i^\sigma = \dot{a}_i$ |
| $H$ elementary abelian, $p$ odd | $\sigma$ may permute cyclic subgroups (Sidki) |

| $H$ | classes | floor $\lvert H \times \dot{H} \rvert$ | orders of $\chi(H, \sigma)$ | ceiling $\lvert \chi(H) \rvert$ |
|---|---|---|---|---|
| $C_3 \times C_3$ | 26 | $3^4$ | $3^4$ (23 classes), $3^5$ (3) | $3^5$ |
| $C_3 \times C_3 \times C_3$ | 252 | $3^6$ | $3^6$ (235), $3^7$ (11), $3^8$ (5), $3^9$ (1) | $3^9$ |
| $C_2 \times C_2 \times C_2$ | 4 | $2^6$ | $2^8$, $2^{10}$ | $2^{10}$ |
| $C_2 \times C_2 \times C_2 \times C_2$ | 3374 | $2^8$ | from $2^9$ to $2^{19}$ | $2^{19}$ |

All are $p$-groups.

► GAP video

# The ceiling, without an isomorphism

$H = C_3 \times C_3 = \langle a, b \rangle$, and $\sigma$ swaps only two images:

$$\sigma\colon\ ab \mapsto \dot{a}^2\dot{b}^2, \quad a^2b^2 \mapsto \dot{a}\dot{b},$$
$$x \mapsto \dot{x} \text{ otherwise}$$

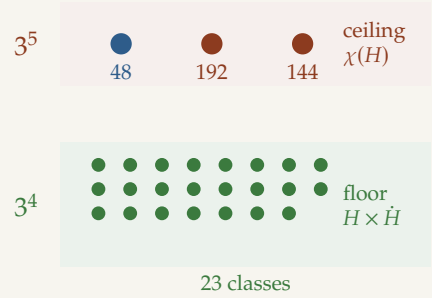
Not an isomorphism: $\sigma(a)\sigma(b) = \dot{a}\dot{b} \neq \sigma(ab)$.

$\{\dot{x} : [ab, \dot{x}] = 1\}$ is a subgroup, so

$$[ab, (\dot{a}\dot{b})^2] = 1 \iff [ab, \dot{a}\dot{b}] = 1.$$

same relations: $\chi(H, \sigma) = \chi(H)$ `[243,65]`

**All 26 classes of $C_3 \times C_3$**

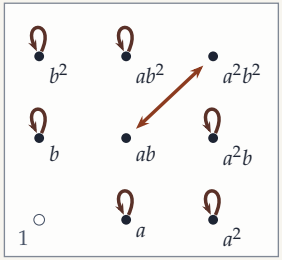



one dot per class; the number = bijections in it

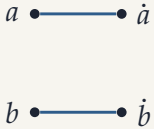
- 48 = $|\mathrm{Aut}(H)|$, the isomorphisms
- 192 + 144 = 336 non-isomorphisms

# Two transpositions in $C_3 \times C_3$

**inside a cyclic subgroup:** $ab \leftrightarrow a^2b^2$




$\sigma$ seen from above

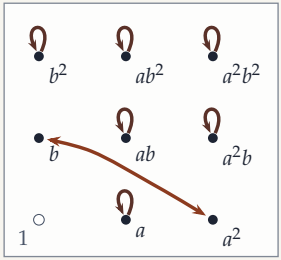



$[a_i, \dot{a}_j] = 1$

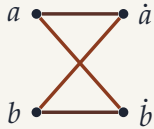
$[ab, (\dot{a}\dot{b})^2] = 1 \iff [ab, \dot{a}\dot{b}] = 1$: the swap is a loop again.

$\chi(H, \sigma) = \chi(H)$ `[243,65]`

**across cyclic subgroups:** $a^2 \leftrightarrow b$




$\sigma$ seen from above




$[a_i, \dot{a}_j] = 1$

$[a^2, \dot{b}] = 1 \Rightarrow [a, \dot{b}] = 1$: a new edge, the graph is complete.

$\chi(H, \sigma) = H \times \dot{H}$ `[81,15]` (video)

Neither $\sigma$ is an isomorphism (GAP). Same kind of move, opposite ends of the band.

# The floor is not always reached

| $H$ | bijections | classes | orders of $\chi(H,\sigma)$ | $\lvert H \times \dot{H}\rvert$ |
|---|---|---|---|---|
| $C_2 \times C_2$ | $3! = 6$ | 1 | $2^5$ | $2^4$ |
| $C_2 \times C_2 \times C_2$ | $7! = 5040$ | 4 | $2^{10}$, $2^{10}$, $2^8$, $2^8$ | $2^6$ |
| $C_3 \times C_3$ | $8! = 40320$ | 26 | $3^5$ (3), $3^4$ (23) | $3^4$ |

- $C_2 \times C_2$: the six bijections are all equivalent to the isomorphism: $\chi(H,\sigma) \cong \chi(H)$.
- $C_2 \times C_2 \times C_2$: the two groups of order $2^8$ differ `[256,55805]`, `[256,55960]`.
- $C_2^n$: $\langle a_i \rangle \setminus \{1\} = \{a_i\}$, so at most $n$ of the $n^2$ pairs $[a_i, \dot{a}_j]$ are reached directly.
- Not the prime 2: for $D_8$ and $Q_8$ the floor *is* reached (76 of 90, 12 of 16 classes).

When is the floor $H \times \dot{H}$ reached?

► video

# $p > n$: the direct product

Let $H = C_p^n = \langle a_1, \ldots, a_n \rangle$.

## Theorem

If $p > n$ and $\sigma(a_i^j) = \dot{a}_j^i$ for $1 \le i, j \le n$ (any extension), then $\chi(H, \sigma) = H \times \dot{H}$.

Key point: in $\chi(H, \sigma)$, $\{h \in H : [h, \dot{y}] = 1\}$ is a subgroup, so

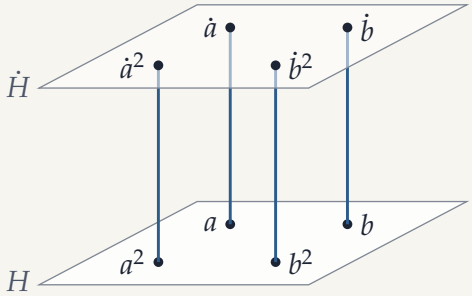
$$[x^i, \dot{y}^j] = 1, \quad p \nmid ij \quad \Longrightarrow \quad [x, \dot{y}] = 1.$$

## Conjecture (chain construction)

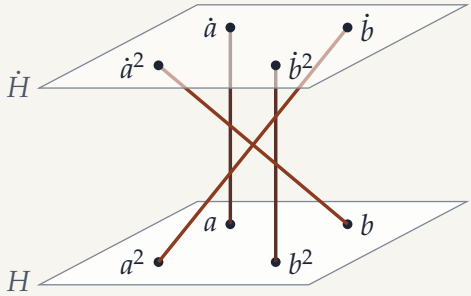
If $p$ is odd, $n \le 2p - 1$ and $\sigma(a_i^k) = \dot{a}_{i+k}^k$ ($1 \le k \le p - 1$, indices mod $n$), identity elsewhere, then $\chi(H, \sigma) = H \times \dot{H}$.

Random $\sigma$ for $C_3^6$, $C_3^7$, $C_3^8$ always gave $H \times \dot{H}$: a generic bijection kills $[H, \dot{H}]$.

# $H = C_3 \times C_3$: two copies of $H$



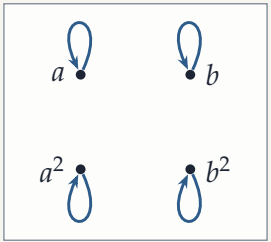

$\chi(H)$: $[x, \dot{x}] = 1$




$\chi(H, \sigma)$: $[x, x^\sigma] = 1$

$\sigma : a \mapsto \dot{a},\ \ a^2 \mapsto \dot{b},\ \ b \mapsto \dot{a}^2,\ \ b^2 \mapsto \dot{b}^2$ (mixed elements fixed: the transposition $a^2 \leftrightarrow b$)

# $H = C_3 \times C_3$: seen from above





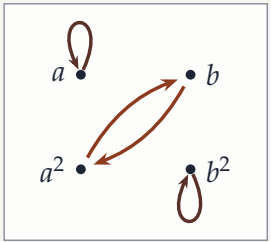


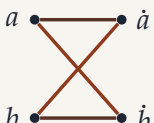
$H \equiv \dot{H}$

$\sigma$ seen from above

$[a_i, \dot{a}_j] = 1$

$\chi(H)$




$H \equiv \dot{H}$

$\sigma$ seen from above

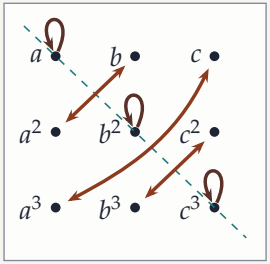
$[a_i, \dot{a}_j] = 1$

$\chi(H, \sigma) = H \times \dot{H}$

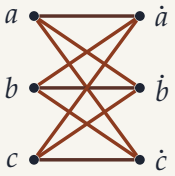
loop at $x$: $\sigma(x) = \dot{x}$ arrow $x \to y$: $\sigma(x) = \dot{y}$ edge $x$ — $\dot{y}$: $[x, \dot{y}] = 1$ is forced

For $\sigma$ the graph is complete bipartite.

# The next case: $H = C_5 \times C_5 \times C_5$




$\sigma$ seen from above




$[a_i, \dot{a}_j] = 1$

$\sigma(a_i^j) = \dot{a}_j^i$: the transpose $\quad K_{3,3}$: $\chi(H, \sigma) = H \times \dot{H}$

$p = 5 > n = 3$. The other $5^3 - 1 - 9 = 115$ elements of $H \setminus \{1\}$ go anywhere.



Graduation, with my classmates



PhD defense, UnB, 2016



PhD defense, UnB, 2017



Research visit, Goiânia, 2023




Master's defense, IME/UFG, 2023

# Happy birthday, Professor Noraí!

Thank you.

► photos