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Q-QUADRATIC CONVERGENCE ON NEWTON'S METHOD FROM DATA AT ONE POINT

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Abstract: Smale's Theorem on Newton's Method for analytic systems provides existence of a solution and R-quadratic convergence of the method from data at one point. In this paper, we proof that Newton Method under Smale's hypothesis is Q-quadratic convergent and as a consequence, we deduce an error estimate.

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1. Introduction

By "solving" $F(x) = 0$ we shall understand to get an "approximated solution", i.e., to get a point x_0 such that Newton Method for solving $F(x) = 0$, with starting point x_0 generates a sequence that converges to a solution. In particular, an "approximated solution" implies the existence of a solution. The most important question is to decide whether a given point is an approximated solution. Smale in [4] has given conditions under which a point is an approximated solution using only the information available at the starting point.

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The aim of this paper is to prove the q-quadratic convergence of Newton Method under Smale's conditions, which is a new result, once up to now only r-quadratic convergence whas proved. As a consequence, we deduce an error estimate.

2. Auxiliary Results

Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces and $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be an analytic map, where $x_0 \in \mathbb{E}$ and $D_r(x_0) = \{x \in \mathbb{E} : ||x - x_0|| \leq r\}$. The derivative of f at $x \in D_r(x_0)$ will be denoted by $Df(x)$ (and the higher order derivatives by $D^k f(x)$). Newton Method for solving

$$f(x) = 0 \tag{1}$$

generates the sequence $\{x_n\}$ by the iterative process

$$x_n = x_{n-1} - Df(x_{n-1})^{-1} f(x_{n-1}) \tag{2}$$

provided that, for all $n \geq 1$, $Df(x_k)^{-1}$ exists. In [4], Smale studied the Newton Method in this context and deduced consequences from data at a single point, but only R-quadratic convergence, see Ortega et al [1], is obtained. As in Shub at al [3], we use the same notation to obtain Q-quadratic convergence, see Ortega at al [1], of Newton Method for this context.

For every point $x \in D_r(x_0)$ define

$$\beta(f,x) = ||Df(x)^{-1}f(x)||, \quad \gamma(f,x) = \sup_{k>1} \left|\left| \frac{Df(x)^{-1}D^k f(x)}{k!} \right|\right|^{\frac{1}{k-1}} \tag{3}$$

and if $Df(x)^{-1}$ does not exist, define $\beta(f,x) = \infty$ and $\gamma(f,x) = \infty$. Now define, $\alpha(f,x) = \beta(f,x)\gamma(f,x)$.

The following expressions play an important role in the next results:

$$\tau(\alpha) = \frac{(1+\alpha) - \sqrt{(1+\alpha)^2 - 8\alpha}}{4}, \quad \text{for } 0 \leq \alpha \leq 3 - 2\sqrt{2}, \tag{4}$$

$$\alpha_0 = \frac{1}{4}(13 - 3\sqrt{17}). \tag{5}$$

A point x_0 is called an *approximated zero* of f, if the sequence $\{x_n\}$, generated by (2), is well defined and satisfies:

$$||x_n - x_{n-1}|| \leq \left(\frac{1}{2}\right)^{2^{n-1}-1} ||x_1 - x_0|| \tag{6}$$

for all $n \geq 1$. The next result gives us conditions under which a point x_0 is an approximated zero, for the proof see Shub at al [3].

Theorem 1. *Let $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be analytic map, $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ and $r \geq \frac{\tau(\alpha)}{\gamma}$. Then if $\alpha \leq \alpha_0$, the Newton iterates $x_1, x_2, \ldots$ are defined well, converge to $\zeta \in D_r(x_0)$ with $f(\zeta) = 0$ and for all $n \geq 1$*

$$||x_n - x_{n-1}|| \leq \left(\frac{1}{2}\right)^{2^{n-1}-1} ||x_1 - x_0||\,. \tag{7}$$

Moreover, $||\zeta - x_0|| \leq \frac{\tau(\alpha)}{\gamma}$, and $||\zeta - x_1|| \leq \frac{\tau(\alpha)-\alpha}{\gamma}$.

Theorem 1 implies that the sequence $\{x_k\}$ satisfies

$$||x_n - \zeta|| \leq \left(\frac{1}{2}\right)^{2^n} ||x_1 - x_0|| K\,, \tag{8}$$

where $K = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{2^{n-1}-1}$, see Smale [4], and this inequality signifies that the sequence $\{x_k\}$ has convergence R-quadratic.

Now, for each $\beta, \gamma > 0$, define

$$h_{\beta,\gamma}(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}. \tag{9}$$

Let $\alpha = \beta\gamma$ satisfy $(\alpha + 1) - 8\alpha > 0$ or equivalently $0 < \alpha < 3 - 2\sqrt{2}$. Then $h_{\beta,\gamma}(t) = 0$ has two distinct real positive roots, the smaller root is

$$\frac{\tau(\alpha)}{\gamma} = \frac{(1+\alpha) - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}. \tag{10}$$

Moreover $d^2 h_{\beta,\gamma}/dt^2(t) > 0$ as long as $0 < t < \frac{1}{\gamma}$ which implies that $h_{\beta,\gamma}$ is convex in this interval. Thus Newton Method, to solving $h_{\beta,\gamma}(t) = 0$, starting at $t_0 = 0$ generates the monotone sequence $\{t_n\}$ which converges to $\frac{\tau(\alpha)}{\gamma}$.

Theorem 2. *(Domination Theorem) Let $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be analytic map, $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ and suppose $r \geq \frac{\tau(\alpha)}{\gamma}$ and $\alpha \leq \alpha_0$. These values of β, α define $h_{\beta,\gamma}$ and the sequence $\{t_k\}$. Then*

$$||x_n - x_{n-1}|| \leq t_n - t_{n-1}, \quad n = 1, 2, \ldots, \tag{11}$$

where $\{x_n\}$ is the Newton sequence of f starting at x_0.

It follows from (11) that

$$||x_n - x_0|| \leq t_n, \quad n = 1, 2, \ldots \tag{12}$$

and this implies that $\{x_n\} \subset D_{\frac{\tau(\alpha)}{\gamma}}(x_0)$.

Let

$$\psi(u) = 2u^2 - 4u + 1, \quad 0 \leq u \leq 1 - \frac{\sqrt{2}}{2}, \tag{13}$$

so that $0 \leq \psi(u) \leq 1$.

Lemma 1. *Let $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be analytic map and let $\gamma = \gamma(f, x_0)$. If $x \in D_r(x_0)$ with $\psi(u) > 0$, where $u = ||x - x_0||\gamma$, then*
(1) $Df(x)$ is invertible;
(2) $||Df(x)^{-1}Df(x_0)|| \leq \frac{(1-u)^2}{\psi(u)}$.

Proof. See Shub at al [3], Lemma 3, p. 476. □

We observe that

$$\frac{(1-u)^2}{\psi(u)} = -\frac{1}{h'_{\beta,\gamma}(u)}. \tag{14}$$

Since $h'_{\beta,\gamma}$ is monotone, then from (14) it follows that, for all $\alpha \leq \alpha_0$, and $x \in D_{\frac{\tau(\alpha)}{\gamma}}(x_0) = \{x \in E : ||x - x_0|| \leq \frac{\tau(\alpha)}{\gamma}\}$,

$$\frac{(1 - ||x - x_0||\gamma)^2}{\psi(||x - x_0||\gamma)} \leq \frac{(1 - \tau(\alpha))^2}{\psi(\tau(\alpha))}. \tag{15}$$

3. *Q*-Quadratic Convergence

This is the main section. Here we prove that, under Smale's Conditions, the sequence generated by Newton Method *Q*-quadratically converges and as a consequence, we deduce an error estimate.

Lemma 2. *(Lemma of Calculus) Let $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be continuous, differentiable in the interior $D_r^o(x_0)$ of $D_r(x_0)$ and $Df(z_0)$ be non-singular. Suppose that, for all $x, x' \in D_r(x_0)$*

$$\left|\left|Df(x_0)^{-1}(Df(x) - Df(x'))\right|\right| \leq L\left|\left|x' - x\right|\right|.$$

If $x \in D_r^o(x_0)$, $v \in \mathbb{E}$, $t \in R$ and $x + tv \in D_r(x_0)$, then

$$f(x + tv) = f(x) + tDf(x)\, v + R(t) \quad \text{with} \quad ||Df(x_0)^{-1}R(t)|| \leq \frac{L}{2}t^2||v||^2.$$

Proof. Follows from "Fundamental Theorem of Calculus". □

Lemma 3. *Let* $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ *be analytic map,* $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ *and* $r > \frac{\tau(\alpha)}{\gamma}$. *If* $\alpha \leq \alpha_0$ *then, for all* $x, x' \in D_{\frac{\tau(\alpha)}{\gamma}}(x_0)$

$$||Df(x_0)^{-1}(Df(x') - Df(x))|| \leq \frac{2\gamma}{(1-\tau(\alpha))^3}||x' - x||. \tag{16}$$

Proof. Let $w \in D_{\frac{\tau(\alpha)}{\gamma}}(x_0)$. Note that

$$Df(x_0)^{-1}D^2f(w) = \sum_0^\infty \frac{1}{k!}Df(x_0)^{-1}D^{k+2}f(x_0)(w - x_0)^k. \tag{17}$$

Since $\alpha \leq \alpha_0$ we have that $\gamma||w - x_0|| \leq \tau(\alpha) < 1$, thus from (17) it follows that

$$\begin{aligned} |Df(x_0)^{-1}D^2F(w)|| &\leq \gamma \sum_0^\infty (k+2)(k+1)(\gamma||w - x_0||)^k \\ &= \frac{2\gamma}{(1-\gamma||w - x_0||)^3} \\ &\leq \frac{2\gamma}{(1-\tau(\alpha))^3}. \end{aligned} \tag{18}$$

But since

$$||Df(x_0)^{-1}(Df(x) - Df(x'))|| \leq \sup_{w \in D_r(x_0)} ||Df(x_0)^{-1}D^2f(w)||\,||x' - x||,$$

it follows from (18) the statement of the lemma. □

Theorem 3. *Let* $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ *be analytic map,* $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ *and* $r > \frac{\tau(\alpha)}{\gamma}$. *Then if* $\alpha \leq \alpha_0$, *the Newton iterates* $x_1, x_2, \ldots$ *are defined well, converge to* $\zeta \in D_r(x_0)$ *with* $f(\zeta) = 0$ *and there exists a constant* $M = M(x_0)$ *such that*

$$||x_{n+1} - \zeta|| \leq M||x_n - \zeta||^2$$

for all $n \geq 1$.

Proof. Since $\alpha \leq \alpha_0$ from Theorem 1 it follows that x_0 is an approximated zero of f, then the sequence $\{x_n\}$ converges to ζ, where $f(\zeta) = 0$ and from equation (12) the sequence $\{x_n\} \subset D_{\frac{\tau(\alpha)}{\gamma}}(x_0)$. Furthermore, from Lemma 1 and (15), it follows that

$$||Df(x_n)^{-1}Df(x_0)|| \leq \frac{(1-\tau(\alpha))^2}{\psi(\tau(\alpha))}. \tag{19}$$

Now from Lemma 2 and Lemma 3 it follows that

$$||DF(x_0)^{-1}R_n|| \leq \frac{2\gamma}{2(1-\tau(\alpha))^3}||x_n - \zeta||^2, \tag{20}$$

where

$$f(\zeta) = f(x_n) + Df(x_n)\,(\zeta - x_n) + R_n. \tag{21}$$

Thus the inequalities (19), (20) and $f(\zeta) = 0$ imply that

$$\begin{aligned} ||x_{n+1} - \zeta|| &\leq ||Df(x_n)^{-1}Df(x_0)||\,||Df(x_0)^{-1}R_n|| \\ &\leq M||x_n - \zeta||^2, \end{aligned} \tag{22}$$

where $M = \dfrac{(1-\tau(\alpha))^2}{\psi(\tau(\alpha))}\dfrac{2\gamma}{2(1-\tau(\alpha))^3} = \dfrac{\gamma}{\psi(\tau(\alpha))(1-\tau(\alpha))}$. □

Theorem 4. *Let $\{x_n\}$ be a sequence in Banach space $\mathbb{E}$, convergent to ζ such that*

$$||x_{n+1} - \zeta|| \leq a||x_n - \zeta||^2 \tag{23}$$

for all n and a positive constant a. If $\mu < 1/4$, $a||x_{n+1} - x_n|| < \mu$ and $a||\zeta - x_n|| \leq \frac{2}{1+\sqrt{1-4\mu}}$, then

$$\frac{2}{1+\sqrt{1+4\mu}} \leq \frac{||\zeta - x_n||}{||x_{n+1} - x_n||} \leq \frac{2}{1+\sqrt{1-4\mu}}.$$

Proof. See Ostrowski [2], pp. 372, 373. □

From Theorem 3 and Theorem 4 we obtain the following theorem.

Theorem 5. *Let $f\colon D_r(x_0) \longrightarrow \mathbb{F}$ be analytic map, $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ and $r \geq \frac{\tau(\alpha)}{\gamma}$. Then if $\alpha \leq \alpha_0$, the Newton iterates $x_1, x_2, \ldots$ are defined well, converge to $\zeta \in D_r(x_0)$ with $f(\zeta) = 0$ and there exists a constant $\mu \leq .115146$ such that*

$$\frac{2}{1+\sqrt{1+4\mu}} \leq \frac{||\zeta - x_n||}{||x_{n+1} - x_n||} \leq \frac{2}{1+\sqrt{1-4\mu}}, \tag{24}$$

for all $n \geq 2$.

Proof. From Theorem 3 follows that $\{x_n\}$ satisfies (23) with $a = M$. Define $\mu = \frac{\alpha_0}{8\psi(\tau(\alpha_0))(1-\tau(\alpha_0))} \leq .115146 < 1/4$, where the functions τ and ψ were defined respectively in (10) and (13). Now by (7)

$$M||x_{n+1} - x_n|| \leq \frac{2M\beta}{2^{2^n}} \leq \frac{\alpha}{8\psi(\tau(\alpha))(1-\tau(\alpha))} \leq \mu.$$

For $n > 2$, and by (8) $\|\zeta - x_n\| \le \frac{K\beta}{2^{2^n-1}}$, where $K \le \frac{7}{4}$, we have

$$M\|x_* - x_n\| \le \frac{7\alpha}{16\psi(\tau(\alpha))(1-\tau(\alpha))} \le \frac{7\mu}{2} \le \frac{1+\sqrt{1-4\mu}}{2}.$$

Thus the statement of the theorem follows from Theorem 4. □

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