JUMP BIFURCATIONS IN SOME DEGENERATE PLANAR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH THREE ZONES

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ABSTRACT. We consider continuous piecewise-linear differential systems with three zones where the central one is degenerate, that is, the determinant of its linear part vanishes. By moving one parameter which is associated to the equilibrium position, we detect some new bifurcations exhibiting jump transitions both in the equilibrium location and in the appearance of limit cycles. In particular, we introduce the *scabbard bifurcation*, characterized by the birth of a limit cycle from a continuum of equilibrium points.

1. Introduction and statement of main results

The family of piecewise linear differential systems has become an important class of differential systems, due to its capability to model a large number of engineering problems, see [2, 3, 18, 25] and references therein, as well as models from mathematical biology, see [8, 26, 27]. Despite of its seeming simplicity, there still are unsolved problems regarding stability and bifurcation issues.

In the case of planar systems with two linearity zones separated by a straight line, a lot of effort has been devoted to characterize the maximal number of limit cycles in the discontinuous setting [1, 5, 6, 11, 15, 16, 17], since the continuous case was already solved in [13], see also [24]. However, keeping the continuity of the vector field and dealing even with problems in low-dimensional phase spaces, the study of their dynamics is not completely done.

In this work, we want to clarify some bifurcation phenomena that can appear in planar continuous piecewise linear (CPWL) differential systems with three zones, without any special symmetry conditions but under a specific degeneracy. In particular, we consider the consequences

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of the vanishing of the determinant for the Jacobian matrix of the central zone. As will be shown, this hypothesis leads to a discontinuous behavior in the evolution of the equilibrium point with respect to the selected bifurcation parameter; this fact is rather counter-intuitive as long as the vector field depends continuously on such parameter.

Furthermore, regarding dynamic bifurcations, we reproduce some boundary equilibrium bifurcations leading to limit cycles. In particular, we find:

– an explosive generation of a limit cycle from a continuum of homoclinic and heteroclinic connections, similar to the one studied in [9];
– the generation of small limit cycles that growth linearly with the bifurcation parameter, as in [24]; and
– we also encounter some specific bifurcations, as the introduced *scabbard bifurcation*, characterized by the birth of a limit cycle from a continuum of equilibrium points which, up to the best of our knowledge, has not been reported in the literature.

We focus our attention, as accurate models in some interesting applications, on piecewise linear differential systems have three different linearity regions separated by parallel straight lines, which can be assumed without loss of generality to be the lines $x = -1$ and $x = 1$, see [7]. Thus we have three regions of linearity, namely

$$S_L = \{(x, y) \in \mathbb{R}^2 : x < -1\}, \quad S_C = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\}$$

and

$$S_R = \{(x, y) \in \mathbb{R}^2 : x > 1\},$$

separated by the straight lines

$$\Sigma_\pm = \{(x, y) \in \mathbb{R}^2 : x = \pm 1\}.$$

Furthermore, it is rather usual for these systems to exhibit only one equilibrium point, whose position can be controlled by moving one parameter. This happens in particular when all the determinants of the involved linear parts are positive. Then, under these generic assumptions, see [7], and denoting with α the main bifurcation parameter, our CPWL systems can be written in the Liénard form

$$\begin{aligned} \dot{x} &= F(x) - y, \\ \dot{y} &= g(x) - \alpha, \end{aligned} \tag{1}$$

where the dot denotes derivatives with respect to a time variable τ,

$$F(x) = \begin{cases} t_R(x-1) + t_C, & \text{if } x \geq 1, \\ t_C x, & \text{if } |x| \leq 1, \\ t_L(x+1) - t_C, & \text{if } x \leq -1, \end{cases} \tag{2}$$

and

$$g(x) = \begin{cases} d_R(x-1) + d_C, & \text{if } x \geq 1, \\ d_C x, & \text{if } |x| \leq 1, \\ d_L(x+1) - d_C, & \text{if } x \leq -1. \end{cases}$$

Here, t_Z and d_Z with $Z \in \{L, C, R\}$ denote the trace and determinant in each linear zone.

Note that the above formulation includes as particular cases the following ones. If $t_C = t_L$ and $d_C = d_L$ then we have a system with only two different linearity zones, thoroughly analyzed in [13]. If $t_R = t_L$, $d_R = d_L$ and $\alpha = 0$, then we have a symmetric system with three different linearity zones, thoroughly analyzed in [12]. Non-symmetric systems were considered in [20, 21]. A simpler case included into the previous formulation was studied in [19], where authors consider the non-generic situation $d_R > 0$, $t_R = 0$ and $d_C > 0$.

Remark 1. *Note that CPWL systems are Lipschitz and so they satisfy the standard results on existence and uniqueness of solution as well as their continuous dependence respect to initial conditions and parameters. In fact, the solutions are functions of class $\mathcal{C}^1$ and we emphasize that several classical results of the qualitative theory of planar differential systems, see* [10]*, and in particular Poincaré-Bendixson's and Dulac's theorems can be adequately extended to cover these CPWL systems.*

Our initial assumption on the uniqueness of equilibrium point required the determinants in all the three zones to be positive. In this paper we will consider instead a degenerate situation by assuming that the determinant in the central zone vanishes, that is, $d_C = 0$, keeping the original assumptions $d_L, d_R > 0$. This setting arises in a natural way when one wants to analyse certain Petri nets, see for instance [22]. By considering the second equation in (1), equilibrium points should be located at points $(x, y) = (\bar{x}, \bar{y})$, where $\bar{x}$ is any solution of $g(x) = \alpha$ and $\bar{y} = F(\bar{x})$, being now

$$g(x) = \begin{cases} d_R(x-1), & \text{if } x \geq 1, \\ 0, & \text{if } |x| \leq 1, \\ d_L(x+1), & \text{if } x \leq -1. \end{cases} \tag{3}$$

Thus, regarding the equilibrium solutions of system (1)–(3), we can state the first consequence of the above assumptions. The proof of this first result is straightforward and will be omitted.

Lemma 1. *The following statements hold for system* (1)–(3)*.*

(a) *For $\alpha < 0$ the system has only one equilibrium point, which is in the left zone, namely at*

$$e_L = (\bar{x}_L, \bar{y}_L) = \left(-1 + \frac{\alpha}{d_L}, \frac{\alpha t_L}{d_L} - t_C\right).$$

(b) *For $\alpha = 0$ the system has a continuum of non-isolated equilibrium points, which are in the central zone, namely at every point of the segment*

$$E_C = \{(\bar{x}, \bar{y}) : -1 \leq \bar{x} \leq 1, y = t_C \bar{x}\}.$$

(c) *For $\alpha > 0$ the system has only one equilibrium point, which is in the right zone, namely at*

$$e_R = (\bar{x}_R, \bar{y}_R) = \left(1 + \frac{\alpha}{d_R}, \frac{\alpha t_R}{d_R} + t_C\right).$$

It should be noticed that, when α passes through the critical value $\alpha = 0$, the system exhibits a jump transition in the equilibrium position from the left zone to the right one, see Figure 1. This transition can be also associated to a change in the stability and topological type of the equilibrium, depending on the values of the linear invariants t_Z, d_Z of the external zones, where $Z \in \{L, R\}$. Also, as it will be later shown, the transition could be accompanied with the appearance or disappearance of a limit cycle. In this sense, regarding the traces t_L, t_C, t_R of each zone, we know from Bendixson theory that they all cannot have the same sign to allow the existence of limit cycles.

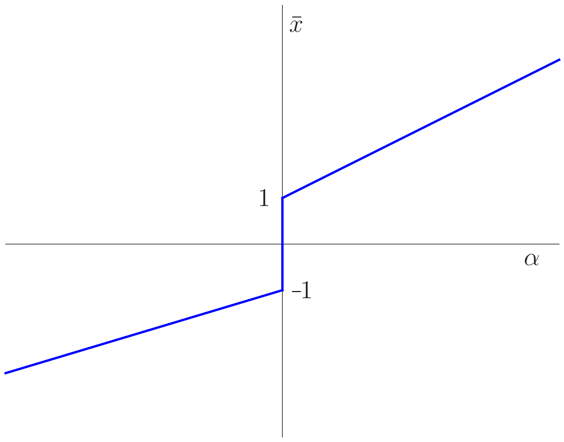

FIGURE 1. The graphic of $\bar{x}$ depending of parameter α.

Since the number of different possibilities is high, here we only consider the cases where $t_L < 0$ and $t_R > 0$, so that the transition is associated to passing from one stable equilibrium point to one unstable one. Once restricted to such case, we must distinguish the different signs of the trace t_C and the different possible dynamics in the external zones (focus or node). To halve the length of our study, we will impose that the dynamics in the right zone is of focus type, that is, we will assume $t_R^2 - 4d_R < 0$.

Whenever we have a focus dynamics in a external zone, it is convenient to introduce some crucial parameters, namely

$$\gamma_Z = \frac{t_Z}{2\omega_Z}, \quad \text{where } \omega_Z = \sqrt{d_Z - \frac{t_Z^2}{4}} \tag{4}$$

and $Z \in \{L, R\}$. Note that γ_Z represents the quotient between real and imaginary parts of the complex eigenvalues of the corresponding linear part. We recall that after a half turn around a focus, that is after a time $\tau = \pi/\omega$, the expansion or contraction factor for the polar radius of solutions is given by

$$e^{\frac{t_Z}{2}\tau} = e^{\pi\gamma_Z}, \text{ where } Z \in \{L, R\},$$

see [14] for more details.

In order to structure all the possible cases, we distinguish two main scenarios: the transition from a stable node to an unstable focus, and the transition from a stable focus to an unstable one. In this last case, we restrict our attention to systems with bounded solutions for positive times, that is, dissipative systems characterized by the condition $\gamma_L + \gamma_R < 0$, see [24].

Before to consider these two scenarios separately, we want to emphasize the possibility of a new specific limit cycle bifurcation in passing from the situation described in Lemma 1 (b) to the one in Lemma 1 (c). In Particular, we consider the bifurcation of a limit cycle from the continuum of equilibria E_C, under the non generic additional condition $t_C = 0$. Due to the shape of the bifurcating limit cycle, resembling a scabbard, see Figure 2, we call this bifurcation as *scabbard bifurcation* and, up to the best of our knowledge, it has been not reported before in the literature, although it already appeared in [23].

Theorem 1 (**Scabbard bifurcation**). *Consider the continuous piecewise linear differential system* (1)–(3)*, where* $d_L, d_R > 0$*,* $t_C = d_C = 0$*,* $t_L < 0$*,* $t_R > 0$*.If both lateral dynamics are of focus type satisfying the condition* $\gamma_L + \gamma_R < 0$*, or we have a stable left node dynamics and a unstable right focus dynamics, then the following statements hold.*

(a) *If* $\alpha = 0$*, then the segment of equilibria* E_C *is the global attractor of the system, although any point in* E_C *is a unstable equilibrium point.*

(b) *If* $\alpha > 0$ *and small, a stable limit-cycle involving the three linearity zones bifurcates from* E_C*, and surrounds the only equilibrium point (unstable focus) predicted by Lemma 1 (c). Such a limit cycle shrinks in height approaching the segment* E_C *and its period tends to infinity as* $\alpha \to 0^+$*.*

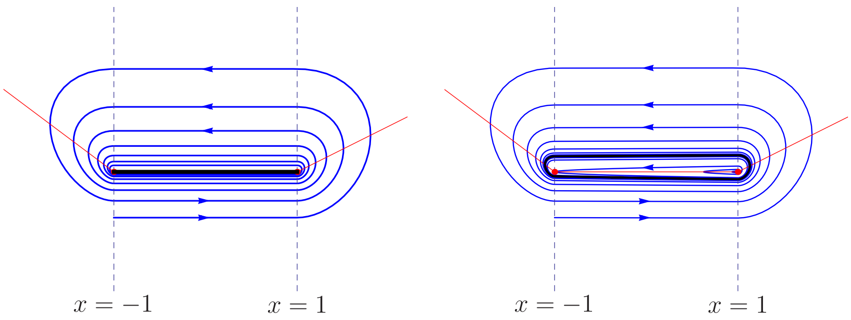

FIGURE 2. The scabbard bifurcation. Here $t_L = -0.75$, $t_C = d_C = 0$, $t_R = 0.5$, $d_L - d_R = 1$. In the left panel $\alpha = 0$, in the right one $\alpha = 0.001$. Note the shape of the limit cycle that bifurcates from the segment of equilibria. The red broken line corresponds with the graph of $y = F(x)$.

This theorem will be proved in Section 2, once we have studied separately the two particular scenarios involved. Note that we only have stated the super-critical version of the bifurcation, as it will be the only case considered in this work.

In what follows, we denote by $T_L = (-1, -t_C)$ and $T_R = (1, t_C)$ the tangency points for the flow of system (1)–(3) with Σ_- and Σ_+, respectively.

1.1. **Transition from a stable node to an unstable focus.** We start by analyzing the critical phase planes, that is, the phase planes for $\alpha = 0$, taking into account the possible different sign of the central trace.

Proposition 1. *Consider the continuous piecewise linear differential system* (1)–(3)*, where* $d_L, d_R > 0$*,* $\alpha = d_C = 0$*,* $t_L < 0$ *with* $t_L^2 - 4d_L \geq 0$ *(left node dynamics) and* $t_R > 0$ *with* $t_R^2 - 4d_R < 0$*. The following statements hold.*

(a) *If $t_C = 0$, then the segment E_C is the global attractor, being all its points unstable equilibrium points; however, the point T_L is the ω-limit set of all the orbits starting at points which are not in the segment E_C.*

(b) *If $t_C > 0$, then all the points of the segment E_C are unstable equilibrium points and there are no periodic orbits. However, there are the following distinguished orbits:*
- *one heteroclinic connection from the point T_R to the point T_L;*
- *one homoclinic orbit H_{T_L} to the point T_L that uses the three zones of linearity, passing through the points $(\pm 1, -t_C)$, and $(\pm 1, t_C(1 + 2\exp(\pi\gamma_R))$, and containing all the points of the segment E_C with $x > -1$ in its interior;*
- *two heteroclinic connections to the point T_L from each point in the segment E_C with $-1 < x < 1$.*

Furthermore, the point T_L is the ω-limit point for all orbits starting not at the segment E_C.

(c) *If $t_C < 0$, then all the points of the segment E_C are stable equilibrium points, but not asymptotically stable points. The segment E_C is a global attractor for the system and so there are no periodic orbits.*

Proposition 1 will be proved in Section 2. Once we know the behavior for $\alpha = 0$, we advance in the next result that the transition from negative to positive values of this parameter always leads to a stable limit cycle. Furthermore, as shown in Theorem 2, the birth of such a limit cycle can have different qualitative behavior, featuring for $t_C \geq 0$ an explosive character.

Proposition 2. *Consider the continuous piecewise linear differential system (1)–(3), where $d_L, d_R > 0$, $d_C = 0$, $t_L < 0$ and $t_R > 0$ with $t_L^2 - 4d_L \geq 0$ and $t_R^2 - 4d_R < 0$. The following statements hold.*

(a) *If $\alpha < 0$, then the equilibrium point e_L is a stable node, being the global attractor for the system.*

(b) *If $\alpha > 0$, then the equilibrium point e_R is an unstable focus surrounded by at least one stable limit cycle.*

Proposition 2 will be proved in Section 2.

When $t_C > 0$, the limit cycle predicted by statement (b) of above proposition tends, as $\alpha \to 0^+$, to the homoclinic orbit H_{T_L} that exists for $\alpha = 0$, see Figure 4. Therefore, in that case, the transition from negative to positive values of the parameter α gives rise to the sudden appearance of a very big limit cycle; this phenomenon has been called a limit cycle super-explosion in [9]. Some similar phenomenon appears when $t_C = 0$, but in this case the limit cycle bifurcates from the segment of equilibria E_C in a scabbard bifurcation. Finally, when $t_C < 0$, we have another specific boundary equilibrium bifurcation at $\alpha = 0$, which has been analyzed in [24].

We state in the sequel our main result for these three cases.

Theorem 2. *Consider the continuous piecewise linear differential system (1)–(3), where $d_L, d_R > 0$, $d_C = 0$, $t_L < 0$ and $t_R > 0$ with $t_L^2 - 4d_L \geq 0$ and $t_R^2 - 4d_R < 0$. The following statements hold.*

- **(a)** *If $t_C < 0$, then a small stable limit cycle bifurcates at $\alpha = 0$ in a boundary equilibrium bifurcation involving only the central and the right zones. Thus, for $\alpha > 0$ there exists a limit cycle whose size growths linearly with the value of α, as long as the limit cycle does not enter the left zone, that is, while it lies in $S_C \cup \Sigma_+ \cup S_R$. There exists a certain value $\alpha_T > 0$ such that the stable limit cycle becomes tangent to Σ_- at T_L for $\alpha = \alpha_T$. For values of α slightly greater than α_T, the limit cycle uses for sure the three linearity zones.*
- **(b)** *If $t_C = 0$, then a stable limit cycle involving the three linearity zones bifurcates from the segment of equilibria E_C in a 'scabbard' bifurcation. Thus, for $\alpha > 0$ there exists a limit cycle which approaches the segment E_C, with a period tending to infinity, as $\alpha \to 0^+$.*
- **(c)** *If $t_C > 0$, then from the homoclinic orbit H_{t_L} that exists for $\alpha = 0$ as predicted in Proposition 1 (b), a stable limit cycle bifurcates for $\alpha > 0$, that is, the limit cycle approaches such homoclinic orbit as $\alpha \to 0^+$.*

Theorem 2 will be proved in Section 2. Although it is not explicitly proved, we conjecture that in all situations of above theorem, system (1)–(3) has only one stable limit cycle.

1.2. **Transition from a stable focus to an unstable focus.** Here, we consider that in both external zones we have dynamics of focus type and several cases can arise depending on the features of the foci. We restrict our attention to the case where the contraction of the left focus dynamics is able to counteract the expansion of the right focus dynamics. This implies that there are no orbits escaping to infinity and so all orbit are bounded in forward time, what is sometimes referred to as dissipative behavior. From [24], this can be guaranteed whenever $\gamma_L + \gamma_R < 0$.

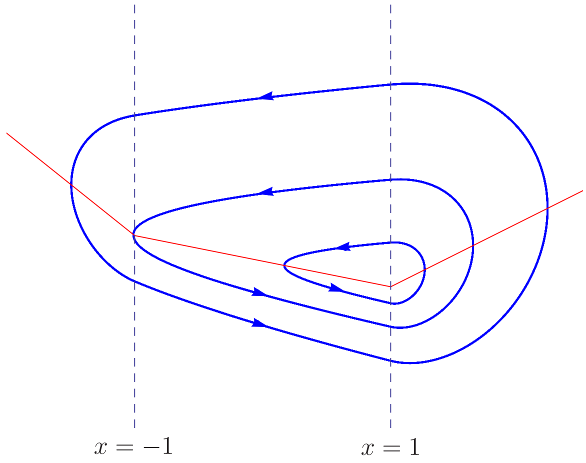

FIGURE 3. The limit cycle for statement (a) in Theorems 2 and 3 taking $\alpha \in \{\alpha_1, \alpha_T, \alpha_2\}$, where $0 < \alpha_1 < \alpha_T < \alpha_2$. Note that the two smaller limit cycles are homothetic.

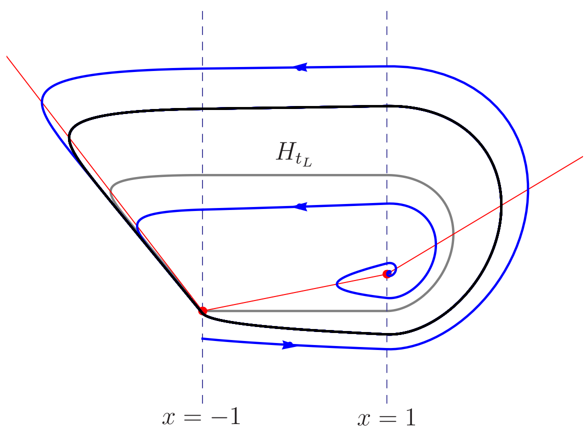

FIGURE 4. Limit cycle bifurcating from homoclinic orbit H_{t_L} in the case Theorem 2 (c). Here, $t_L = -1$, $t_C = 0.2$, $d_C = 0$, $d_R = 1$. The homoclinic orbit H_{t_L} corresponds to $\alpha = 0$, while the limit cycle shown is for $\alpha = 0.03$

As in subsection 1.1, we start by analyzing the critical phase planes, that is, the phase planes for $\alpha = 0$, under the dissipativeness condition, that is, $\gamma_L + \gamma_R < 0$.

Proposition 3. *Consider the continuous piecewise linear differential system* (1)–(3)*, where* $d_L, d_R > 0$*,* $\alpha = d_C = 0$*,* $t_L < 0$ *with* $t_L^2 - 4d_L < 0$ *(left focus dynamics) and* $t_R > 0$ *with* $t_R^2 - 4d_R < 0$ *and* $\gamma_L + \gamma_R < 0$*. The following statements hold.*

- **(a)** *If* $t_C = 0$*, then the segment* E_C *is the global attractor; however, each point of the segment is unstable, since all the orbits starting at points which are not in the segment* E_C *are curves spiraling around* E_C *and approaching it.*
- **(b)** *If* $t_C > 0$*, then there is one stable hyperbolic limit cycle* Γ_S *surrounding the segment of equilibria* E_C *and its intersection points with* Σ_+ *and* Σ_- *are* $(\pm 1, y_0)$ *and* $(\pm 1, y_1)$*, where*

$$y_0 = t_C \frac{1 + 2e^{\pi\gamma_R} + e^{\pi(\gamma_L+\gamma_R)}}{1 - e^{\pi(\gamma_L+\gamma_R)}} > t_C > 0,$$

  and

$$y_1 = -t_C \frac{1 + 2e^{\pi\gamma_L} + e^{\pi(\gamma_L+\gamma_R)}}{1 - e^{\pi(\gamma_L+\gamma_R)}} < -t_C < 0.$$

- **(c)** *If* $t_C < 0$*, then the segment* E_C*, which is constituted by stable, but not asymptotically stable, equilibrium points, is the global attractor and so there are no periodic orbits. All the orbits starting in points which are not in the segment* E_C *have as* ω*-limit set a point on it.*

In the next result, we cannot establish a complete dual result for Proposition 2 because, as shown later, for $\alpha < 0$ there are situations with no limit cycles and other cases with more than one limit cycle.

Proposition 4. *Consider the continuous piecewise linear differential system* (1)–(3)*, where* $d_L, d_R > 0$*,* $d_C = 0$*,* $t_L < 0$ *and* $t_R > 0$ *with* $t_L^2 - 4d_L < 0$*,* $t_R^2 - 4d_R < 0$ *and* $\gamma_L + \gamma_R < 0$*. If* $\alpha > 0$ *then the equilibrium point* e_R *is an unstable focus surrounded by at least one stable limit cycle.*

Now, we state our main result for the focus-focus jump transition, that assures, under certain hypotheses, the existence of at least two limit cycles.

Theorem 3. *Consider the continuous piecewise linear differential system* (1)–(3)*, where* $d_L, d_R > 0$*,* $d_C = 0$*,* $t_L < 0$ *and* $t_R > 0$ *with* $t_L^2 - 4d_L < 0$*,* $t_R^2 - 4d_R < 0$ *and* $\gamma_L + \gamma_R < 0$*. The following statements hold.*

- **(a)** *If* $t_C < 0$*, then a small stable limit cycle bifurcates at* $\alpha = 0$ *in a boundary equilibrium bifurcation involving only the central* *and the right zones. Thus for $\alpha > 0$ there exists a limit cycle whose size growths linearly with the value of α, as long as the limit cycle does not enter the left zone, that is, while it lies in $S_C \cup \Sigma_+ \cup S_R$. There exists a certain value $\alpha_T > 0$ such that the stable limit cycle becomes tangent to Σ_- at T_L for $\alpha = \alpha_T$. For values of α slightly greater than α_T, the limit cycle uses for sure the three linearity zones.*

(b) *If $t_C = 0$, then the system undergoes a "scabbard" bifurcation at $\alpha = 0$; from the segment of equilibria E_C we pass to stable limit cycle involving the three linearity zones for $\alpha > 0$. In other words, this limit cycle approaches the segment E_C, with a period tending to infinity, as $\alpha \to 0^+$.*

(c) *If $t_C > 0$, then a small unstable limit cycle Γ_U bifurcates at $\alpha = 0$ in a boundary equilibrium bifurcation involving only the central and the left zones. Thus, for $\alpha < 0$ with $|\alpha|$ small such a limit cycle growths linearly in size with the value of $|\alpha|$, as long as the limit cycle does not enter the right zone, that is, while it lies in $S_L \cup \Sigma_- \cup S_C$. There exists a certain value $\alpha_T < 0$ such that the limit cycle becomes tangent to Σ_+ at T_R for $\alpha = \alpha_T$. For values of α slightly lower than α_T, the unstable limit cycle uses for sure the three linearity zones.*

Furthermore, for $\alpha_T < \alpha < 0$ there exists at least one stable limit cycle surrounding Γ_U, so that we have at least two limit cycles.

We remark that, statement (c) above assures that for $\alpha_T < \alpha < 0$, there are at least two limit cycles surrounding the stable focus, see Figure 5. We conjecture that there exits a value α^* satisfying $\alpha^* < \alpha_T < 0$ so that for every $\alpha^* < \alpha < \alpha_T$, both limit cycles use the three linearity zones and collide in a semi-stable limit cycle at the value $\alpha = \alpha^*$ to disappear for $\alpha < \alpha^*$. Also, we conjecture that in the situations of statement (a) and (b) of above theorem, system (1)–(3) has only one stable limit cycle.

2. Proof of the main results

We start by showing some elementary facts about some orbits of the system (1)–(3).

Lemma 2. *Under the hypotheses of Proposition 2, and being $\lambda_U \leq \lambda_D < 0$ the two eigenvalues of the left vector field, that is, $\lambda_U + \lambda_D = t_L$*

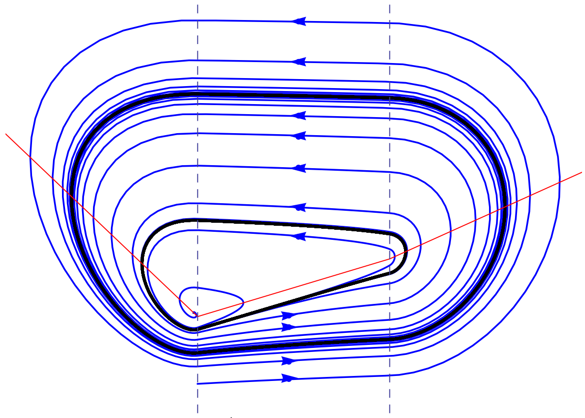

FIGURE 5. Two limit cycles corresponding to Theorem 3 (c). Here $\alpha = -0.04$, $t_L = -0.95$, $t_C = 0.3$, $t_R = 0.45$, $d_L = d_R = 1$, $d_C = 0$.

and $\lambda_U \cdot \lambda_D = d_R$, the two half straight lines

$$y = \lambda_U(x+1) + \frac{\alpha}{\lambda_U} - t_C,$$
$$y = \lambda_D(x+1) + \frac{\alpha}{\lambda_D} - t_C, \tag{5}$$

where $x \leq -1$, are the stable manifolds of the (real or virtual) node, being invariant under the flow. Thus, the intersection point of these lines with Σ_- are the points

$$\left(-1, -t_C + \frac{\alpha}{\lambda_U}\right) \text{ and } \left(-1, -t_C + \frac{\alpha}{\lambda_D}\right).$$

Proof. A straight line $y = mx + b$ is invariant for the left vector field if and only if $\dot{y} = m\dot{x}$, namely

$$d_L(x+1) - \alpha = m[t_L(x+1) - t_C - mx - b],$$

or equivalently

$$m^2 - mt_L + d_L = 0,$$
$$\alpha - d_L + m(t_L - t_C - b) = 0.$$

We see that m must be an eigenvalue for the left vector field, and

$$b = \frac{\alpha - d_L}{m} + t_L - t_C,$$

and then (5) follows easily. □

The following result is stated without any proof, since it is direct.

Lemma 3. *If* $t_C \neq 0$ *and* $d_C = 0$ *then the segment*

$$y = t_C x + \frac{\alpha}{t_C},$$

for $|x| \leq 1$, *is invariant under the flow.*

We define $\Sigma_-^+ = \{(x, y) \in \mathbb{R}^2 : x = -1, y \geq -t_C\}$, $\Sigma_-^- = \{(x, y) \in \mathbb{R}^2 : x = -1, y \leq -t_C\}$, $\Gamma_{T_L}^+$ as the orbit starting at T_L followed forward in time and $\Gamma_{T_L}^-$ as the orbit starting at the same point but followed backwards in time. Clearly, $\Gamma_{T_L}^+ \cap \Gamma_{T_L}^- = T_L$. We will use these two semi-orbits as references for future results. In particular, we can state the following auxiliary lemma, where we only consider $\alpha \neq 0$, since for $\alpha = 0$, both orbits reduce to the point T_L.

Lemma 4. *The following statements hold.*

(a) *For the full orbit* $\Gamma_{T_L} = \Gamma_{T_L}^+ \cup \Gamma_{T_L}^-$ *the point* T_L *represents the maximum value of* x *when* $\alpha < 0$ *and its minimum value when* $\alpha > 0$.

(b) *If* $\alpha < 0$, *then the full orbit* Γ_{T_L} *is totally contained in* $S_L \cup \Sigma_-$, *and its* ω*-limit point is the stable node. Furthermore, such stable node is also the* ω*-limit point for all the orbits starting at* S_L *and above* $\Gamma_{T_L}^-$. *On the contrary, all the orbits starting at* S_L *and below* $\Gamma_{T_L}^-$ *eventually hit* Σ_- *in a point with* $y < -t_C$ *and* $\dot{x} > 0$.

(c) *If* $\alpha > 0$, *then we can define for the orbit* Γ_{T_L} *two notable points* A_1 *and* A_2; A_1 *is the first intersection point of* $\Gamma_{T_L}^-$ *with* Σ_+ *by going backwards in time, while* A_2 *respects the first intersection point of* $\Gamma_{T_L}^+$ *with* Σ_+ *going forward in time. In particular the following cases arise.*

(i) *If* $t_C = 0$, *then the orbit* Γ_{T_L} *satisfy for* $|x| \leq 1$ *the equation*

$$2\alpha(x + 1) = y^2 \tag{6}$$

so that $y_{A_1} = 2\sqrt{\alpha}$, *and* $y_{A_2} = -2\sqrt{\alpha}$.

(ii) *If* $t_C > 0$, *then we have for the points* A_1 *and* A_2 *the inequalities*

$$t_C < y_{A_1} < t_C + \frac{\alpha}{t_C}, \quad y_{A_2} < -2\sqrt{\alpha} < 0 < t_C.$$

(iii) *If* $t_C < 0$, *then we have the inequalities*

$$2\sqrt{\alpha} < y_{A_1}, \quad t_C + \frac{\alpha}{t_C} < y_{A_2} < t_C.$$

Proof. Statement (a) comes easily from the fact that $\ddot{x} = \alpha$ at T_L. Effectively, for $(x, y) \in \Sigma_-$ we have

$$\ddot{x} = t_C\dot{x} - \dot{y} = t_C\dot{x} + \alpha,$$

$$\ddot{x} = t_L\dot{x} - \dot{y} = t_L\dot{x} + \alpha,$$

depending on the vector field selected to make the computation, but as we are computing the derivatives at the line $x = -1$ with $\dot{x} = 0$ there exists continuity for the second derivative and $\ddot{x} = \alpha$.

Statement (b) is a consequence of statement (a), since we have a maximum to respect to x for Γ_{T_L} at T_L. Furthermore, for points in Σ_-^+ we have $\dot{x} < 0$, so that the curve $\Gamma_{T_L}^- \cup \Sigma_-^+$ defines an unbounded positive invariant region whose the stable node is the ω-limit set for all its points. The last assertion is direct, since we are below $\Gamma_{T_L}^-$ and then $\dot{x} > 0$.

To show statement (c), we start by realizing that now we have a minimum with respect to x for Γ_{T_L} at T_L. The existence of the two points A_1 and A_2 comes from the fact that we have in S_C that $\dot{x} < 0$ for $y > t_Cx$ while $\dot{x} > 0$ for $y < t_Cx$ and the slopes tend to be small for $|y|$ sufficiently big. In fact, when $t_C = 0$ we have through an elementary computation the condition (6) and then statement (i) follows. The other statements (ii) and (iii) are also easy to show by taking into account Lemma 3 and that $|t_C + \alpha/t_C| \geq 2\sqrt{\alpha}$ for all $t_C \neq 0$. The Lemma is done.

□

Proof of Theorem 1. Statement (a) comes from Propositions 1(a) and 3(b); and statement (b) comes from Theorems 2(b) and 3(b).

□

Now, we give the proof of Proposition 1.

Proof of Proposition 1. For all the statements, the stability of the points belonging to the segment E_C is clearly determined by the sign of t_C, excepting when $t_C = 0$. In this last case as $\alpha = 0$, from (3) the dynamics from central vector field is given by $\dot{x} = -y$ and $\dot{y} = 0$. Considering the orbits for small, non-vanishing values of y, which are horizontal straight lines, we see that the points in the segment E_C are unstable.

Since from Lemma 2 the left vector field has at least one invariant half straight line with end point T_L, the existence of a periodic orbit can be ruled out for all the situations. Effectively any periodic orbit should surround at least one equilibrium point, but in our case it should surround also the whole segment E_C, what is not possible; otherwise, the periodic orbit should intersect such an invariant half straight line, contradicting the uniqueness of orbits. Therefore there are no periodic orbits.

One can clearly see that, except in the segment E_C, the orbits in the central zone are horizontal segments going from the left to the right for $y < t_C x$ and from the right to the left for $y > t_C x$. Note also that the point T_L is a node, as seen from the left, and similarly, the point T_R is a focus as seen from the right. Thus, we can define a left half-return map $P_L(y)$ for all points $(-1, y)$ in Σ_- with $y \geq -t_C$, so that the orbit starting at $(-1, y)$ comes again to Σ_- at the point $(-1, P_L(y))$; trivially, we see that since $\alpha = 0$, we have

$$P_L(y) = -t_C.$$

On Σ^+ we can define similarly a right half-return map P_R for all the points $(1, y)$ with $y \leq t_C$, and now we have

$$P_R(y) = t_C + (t_C - y)e^{\pi\gamma_R},$$

since the focus is located at the boundary, see [14] for more details.

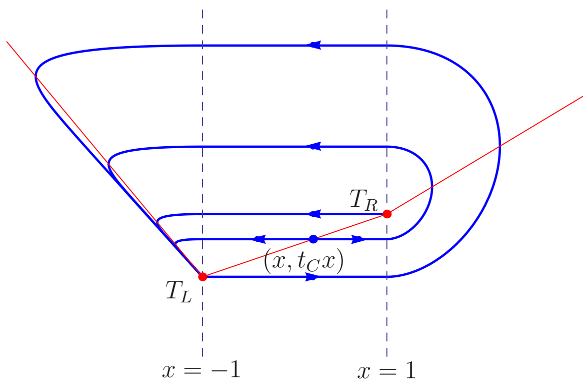

FIGURE 6. The phase plane under hypotheses of Proposition 1 (b), when $\alpha = 0$, $t_L = 1.2$, $t_C = 0.34$, $t_R = 0.6$, $d_L = 0.1$, $d_C = 0$, $d_R = 1$.

To show now the proof of statement (a), it remains to see that as $t_C = 0$ the point t_L is the ω-limit of all the orbits starting at points which are not in the segment E_C. This is evident if we consider that such orbits, after going a half-turn on the right zone, if needed, finally reach Σ_- in a point $(-1, y)$ with $y > 0$. It suffices then to apply the return-map P_L.

To finish the proof of statement (b), we should pay attention only to the existence of specific notable orbits, see Figure 6. Since the final assertion on the point T_L follows a similar reasoning as before. Indeed, take an orbit with initial point (x, t_C) with $|x| < 1$. Again using the fact that in S_C the orbits are horizontal segments, the such an orbit arrives, going forward in time, at the point $(-1, t_C)$ eventually approaching the point T_L. The same orbit, backward in time, approaches the point T_R, and so the existence of the heteroclinic connection is shown.

Take now as initial point $(x, -t_C)$ with $|x| < 1$. Such orbit approaches backward time the point T_L; going forward in time, it arrives at the point $(1, -t_C)$ at Σ_+, and then, after using the map P_R, the orbit will come again to Σ_+ at the point $(1, y_H)$ with

$$y_H = t_C + 2t_C e^{\pi\gamma_R} = t_C(1 + 2e^{\pi\gamma_R}).$$

Then, the orbit will intersect Σ_- at $(-1, y_H)$, to finally approach the point T_L. Thus, the existence of the homoclinic connection is shown.

Let us take now as initial point (ξ, η) where, once fixed $|x| < 1$, we consider $x < \xi < 1$ and $\eta = t_C x$. Thus, the point (ξ, η) is located in the set $\{(x, y) \in \mathbb{R}^2 : |x| < 1 \text{ and } y < t_C x\}$. As before, its orbit, backward in time, approaches the point (x, η) in the segment E_C, but going forward in time we arrive to the point $(1, \eta)$ at Σ_+. Next, after a half-turn around the boundary focus, it comes again to Σ_+ and finally we approach, as before, the point T_L. The same argument, taking now $-1 < \xi < x$ and without any intersection with Σ_+ leads to another heteroclinic orbit from $(x, t_C x)$ to the point T_L. Statement (b) is done.

To show statement (c), we see first that asymptotically stability cannot be achieved since the equilibrium points are not isolated so that, near any equilibrium point, there are orbits whit are not tending to it as time tends to infinity.

To show that the segment E_C is the global attractor, we can start by taking initial points (x, y) with $x < -1$ and $y \geq \lambda_D(x + 1) - t_C$ (above or on the lower invariant half straight line of the node). Clearly the point T_L is the ω-limit point for all those orbits. Keeping $x < -1$ and taking now points below such lower invariant half straight line, we conclude easily that there are three possibilities: the orbit approaches directly a point in the segment E_C; the orbit approaches the segment E_C after a half-turn around the boundary focus at T_R; or finally, the orbit arrives at a point on Σ_- with $y > -t_C$. In this last case, we see that the point T_L is again its ω-limit point and we are done. The proposition is shown. □

Now we are in the point to show Proposition 2

Proof of Proposition 2.
Under hypotheses of statement (a), since $\alpha < 0$ the only equilibrium

point is a node located in S_L. Consequently, we can apply statement (b) of Lemma 4, so that we only need to consider the initial points in $S_C \cup \Sigma_+ \cup S_R$. For such points in $S_C \cup \Sigma_+ \cup S_R$ we have $\dot{y} > 0$, and after doing a half-turn around the virtual focus, if needed, we must arrive to a point in Σ_-^+; then, applying the above reasoning we see that the $\omega-$limit point is again the node. The statement (a) is shown.

To show statement (b), let us start by considering the orbit $\Gamma_{T_L}^+$, which is tangent to Σ_- and goes down to the right up to hitting Σ_+ in the point A_2 with $y_{A_2} < t_C$, see Lemma 4(a). To show the statement, it suffices to consider now the orbit starting at the point $B_U = (-1, \alpha/\lambda_U - t_C)$, that is, the point where the upper invariant half straight line introduced in Lemma 2 intersects Σ_-. Since $\alpha > 0$, the orbit enters S_C and goes down eventually hitting Σ_+ in a point with $y = y_+ < y_{A_2}$.

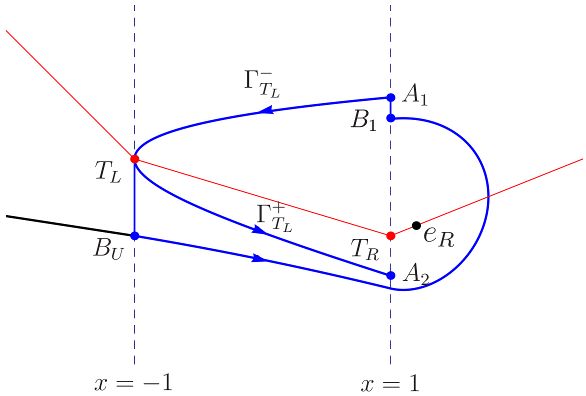

FIGURE 7. The boundary of the positive compact invariant set K_B is composed by the orbit from B_U to B_1, the orbit from A_1 to T_L and the segments B_1A_1 and T_LB_U.

Note that orbits cannot escape to infinity in S_C as long as, for $|y|$ big enough, the slope of orbits approaches zero. If we follow now the orbit of the point $(1, y_+)$ we must surround the unstable focus to hit again Σ_+ in a point B_1 with $y_{B_1} > t_C$. Now the orbit will enter S_C from the right to the left and two possibilities appear. First, let us assume that the point B_1 when the orbit enters S_C is located in Σ_+ and satisfies $y_{B_1} \leq y_{A_1}$. Then the segment B_1A_1, the orbit A_1T_L, the segment T_LB_U and the orbit B_UB_1 form a closed curve that along with their interior defines a compact positive invariant set K_B enclosing the unstable focus, see Figure 7; by Poincaré Bendixson's Theorem, we conclude the existence of one stable limit cycle totally contained in $S_L \cup \Sigma_+ \cup S_R$. Note that in this case of a limit cycle using only two linearity zones, we can assure the uniqueness of the limit cycle by resorting to Theorem 1 of [24].

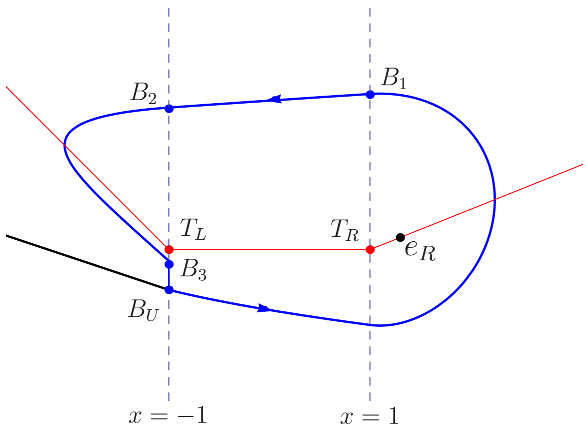

FIGURE 8. The compact positive invariant set K_B for $\alpha > 0$

The second possibility is the case $y_{B_1} > y_{A_1}$. Here we can follow the orbit of the point B_1 in S_C to arrive at a point B_2 in Σ_-^+ entering S_L and hitting again Σ_-, but now in a point $B_3 \in \Sigma_-^-$ with $y_{B_3} > y_{B_U}$. Now the segment B_3B_U and the orbit B_UB_3 form again a closed curve that along with their interior defines a compact positive invariant set K_B, see Figure 8, and we conclude the existence of at least one limit cycle, even we cannot assure its uniqueness. The Proposition is completely shown.

□

Proof of Theorem 2. To show statement (a), we first note that $t_C < 0$ and $t_R > 0$, so that, when $\alpha > 0$ and small, the only equilibrium point predicted in statement (b) of Lemma 1 is near Σ_+ but in S_R. Here, after making the translation $x \to x - 1$ and $y \to y - t_C$, and neglecting for the moment the left zone, we should have a piecewise linear system with only two zones. Then we can directly apply statement (b) of Theorem 1 in [24], by taking there t_L as our t_C and considering the case there with $d_L = 0$, which plays the role of our d_C.

For that bizonal system, it is easy to see that the homogeneous scaling $x \to \alpha X$, $y \to \alpha Y$ gives the system (after suppressing the common factor α)

$$\begin{aligned} \dot{X} &= F(X) - Y, \\ \dot{Y} &= g(X) - 1, \end{aligned} \tag{7}$$

where

$$F(X) = \begin{cases} t_C X, & \text{if } X \leq 0, \\ t_R X, & \text{if } X \geq 0, \end{cases} \tag{8}$$

and

$$g(X) = \begin{cases} 0, & \text{if } X \leq 0, \\ d_R X, & \text{if } X \geq 0. \end{cases} \tag{9}$$

We conclude that, by undoing the rescaling, our limit cycle has a size that is α times the size of the limit cycle that exists for $\alpha = 1$. This argument shows that the size of the limit cycle growths linearly with $\alpha > 0$ and therefore it is born with small size and living just in $S_C \cup S_R$.

If we denote with α_T the value of α corresponding to the limit case in which the limit cycle turns out to be tangent to Σ_- at the point T_L, it is not difficult to see that for $0 < \alpha - \alpha_T << 1$ the limit cycle persists, now using the left zone S_L. Effectively, for $\alpha > \alpha_T$, it is sufficient to consider the orbit with starting point T_L that evolves in $S_C \cup S_R$ coming back to Σ_- in a point A, with $y_A > -t_C$. Indeed, such an orbit tends to approach the bigger limit cycle that should exist if the central zone were prolongated to the left. This orbit determines with the segment $T_L A$ a circuit which is negative invariant. Thus, by using the positive invariant set K_B defined in the proof of statement (b) of Proposition 2, see Figure 8, there is a limit cycle using the three linearity zones. Statement (a) is done.

To show statement (b), we recall from Lemma 4 (c) that the orbit $\Gamma_{T_L}^-$ and $\Gamma_{T_L}^+$ intersect Σ_+ at the points A_1 and A_2, respectively. We claim that the orbit starting at A_2 enters S_R and, after doing a half return around the unstable focus, comes again to Σ_+ at a point A_3 with $y_{A_3} > y_{A_1}$, see Figure 9. It allows us to define a negative invariant compact set K_S. Effectively, if we assume $y_{A_3} \leq y_{A_1}$, then the segment A_3A_1, along with the orbits $\Gamma_{T_L}^-$, $\Gamma_{T_L}^+$ and A_2A_3 should define a compact positive invariant set enclosing the unstable focus. Consequently, by Poincaré-Bendixson's Theorem we should conclude the existence of a stable limit cycle within such a compact set. But this is impossible by the Dulac's criterion as the divergence is positive in S_R and vanishes in S_C. The claim is shown, and as a consequence, the same closed circuit used before turns out to be a compact negative invariant set.

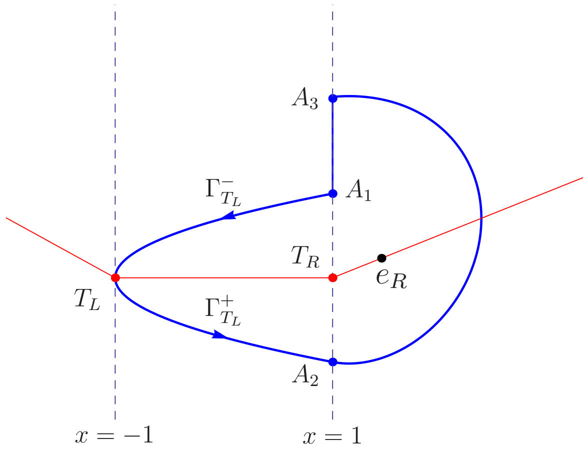

FIGURE 9. The small compact negative invariant set K_S.

Now, taking the point B_U as initial point, we can build exactly as in the proof of statement (b) of Proposition 2 a bigger compact positive invariant set K_B enclosing the above circuit. Thus, the stable limit cycle predicted in Proposition 2 is located between the boundaries of the two compact sets K_S and K_B and so it uses always the three linearity zones after the 'scabbard' bifurcation. Finally, it is not difficult to see that for both compact sets the limit when $\alpha \to 0^+$ is the segment E_C. Moreover, its period is bounded from below by two times the time needed to pass from Σ_- to Σ_+; since $\dot{y} = -\alpha$ an easy easy calculation shows that the necessary time for a single transition is equal to $2/\sqrt{\alpha}$ and so the period tends to infinity as $\alpha \to 0^+$

Statement (c) comes from a similar argument. We use again the point B_U to build as before the big compact positive invariant set K_B. Similarly, we can build a smaller compact negative invariant set K_S. Under our hypotheses, now these two sets have no common limit set when $\alpha \to 0^+$. Of course there is a limit cycle between the boundaries of these two sets K_B and K_S. When $\alpha \to 0^+$, the boundary of K_B tends to the homoclinic orbit H_{T_L} of Proposition 1. However, the boundary of K_S behaves in a different way as it tends to lower right part of H_{T_L} but to the segment E_C plus the segment joining T_R and $(1, y_H)$ for the upper left point. This can be rigorously shown by considering the segment of Lemma 3 which is a upper bound for the semiorbit $\Gamma^-_{T_L}$. Anyway, as the limit cycle approaches the homoclinic orbit H_{T_L} on its lower right part, it also approaches the upper left part of H_{T_L} due to uniqueness of solution of the system. The theorem is done. $\square$

Proof of Proposition 3. We follow a parallel argument to the one in the proof of Proposition 1 taking into account that now the point T_L is a focus. Thus, we have for $y \leq t_C$ the right Poincaré half-return map

$$P_R(y) = t_C + (t_C - y)e^{\pi\gamma_R},$$

as before. We can define for points $(-1, y)$ on Σ_- with $y > -t_C$ the corresponding left half-return map

$$P_L(y) = -t_C - (y + t_C)e^{\pi\gamma_L},$$

and recall that the transitions of orbits within S_C are horizontal paths.

To show statement (a), the instability of all the points in the segment E_C follows from the same argument than in Proposition 1 (a). The global attraction of the segment E_C and the non-existence of periodic orbits comes directly from the fact that when $\alpha = 0$, we have

$$P(y) = e^{\pi(\gamma_L+\gamma_R)}y,$$

which is a contractive map, since $\exp(\pi(\gamma_L + \gamma_R)) < 1$.

For the statement (b), a direct computation gives for $y \geq -t_C$, $P_L(y) \leq -t_C$ and $P_R(P_L(y))$ is well defined, so that

$$P(y) = P_R(P_L(y)) = t_C(1 + 2e^{\pi\gamma_R} + e^{\pi(\gamma_L+\gamma_R)}) + ye^{\pi(\gamma_L+\gamma_L)}. \tag{10}$$

Now, solving for $P(y) = y$ the only solution is the value y_0 given in the statement. The value of y_1 follows straightforward.

In statement (c), we can rule out the existence of periodic orbits as the only possibility should be associated to the previous computed value of y_0 but now we have $y_0 < t_C < -t_C$ and so it is out of the valid domain of the map P_L. Regarding the stability of points in the segment E_C, it comes as in Proposition 1 (c).

□

Proof of Proposition 4. Since $\alpha > 0$ we have an unstable focus at $(\bar{x}_R, \bar{y}_R)$, see Lemma 1 (b). Without computing explicitly the complete Poincaré map, we start by taking as Poincaré section the vertical line $x = \bar{x}_R > 1$. Then, for small values of $\hat{y} = y - \bar{y}_R > 0$, as long as the orbits around the the focus do not use the region S_C, we can write

$$P(\hat{y}) = e^{2\pi\gamma_R}\hat{y} > \hat{y},$$

where we pass from the point $(\bar{x}_R, y) = (\bar{x}_R, \bar{y}_R + \hat{y})$ to the point $(\bar{x}_R, \bar{y}_R + P(\hat{y}))$, after a complete turn around the focus. Thus we have $P'(0) = e^{2\pi\gamma_R} > 1$.

For sufficiently big values of $\hat{y} > 0$, the orbit starting at $(\bar{x}_R, \hat{y} + \bar{y}_R)$ will go around the focus, and then it will enter S_C and also it will lives in S_L by doing a half turn around the point T_L, to go back to our

Poincaré section after using again S_C. Furthermore, the two transitions on S_C and in the band $B = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq \bar{x}_R\}$ of such an orbit approach horizontal paths as $y \to \infty$, while the flight times on S_L and $S_R \backslash B$ tends to π/ω_L and π/ω_R, respectively. Thus, it is straightforward to see that the asymptotic behavior of our Poincaré map is such that

$$\lim_{\hat{y}\to\infty} \frac{P(\hat{y})}{e^{\pi(\gamma_L+\gamma_R)}\hat{y}} = 1,$$

so that

$$\lim_{\hat{y}\to\infty} P'(\hat{y}) = \lim_{\hat{y}\to\infty} \frac{P(\hat{y})}{\hat{y}} = e^{\pi(\gamma_L+\gamma_R)} < 1.$$

We conclude that the graph of P has at least one intersection with the diagonal, and so the system must have at least one periodic orbit. □

Proof of Theorem 3. Statement (a) can be shown exactly as in Theorem 2 (a), but now we need a different compact positive invariant set K_B. This set can be built easily by considering that the point at infinity is repulsive; thus, enough to take an orbit starting at Σ_- in a new point $(-1, y)$ with $y < 0$ and $|y|$ big enough, instead B_U.

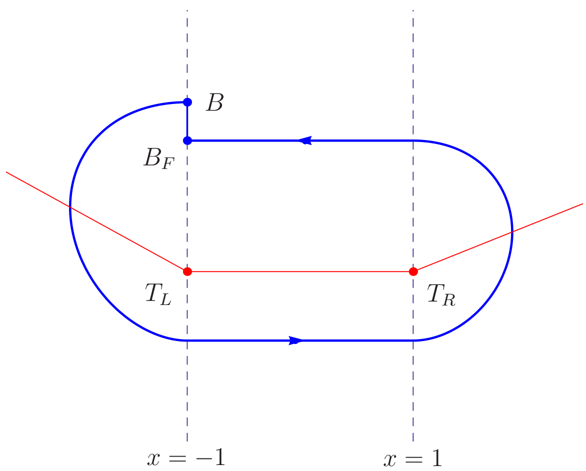

FIGURE 10. The compact positive invariant set K_B in the case focus-focus when $t_C = 0$ and $\alpha = 0$.

To show statement (b) we note that when $\alpha > 0$ we can build the negative invariant compact set K_S as in the proof of Theorem 2 (b). We emphasize that this compact set K_S can be chosen as smaller as one wants, by selecting a small value of $\alpha > 0$, see Figure 9.

On the other hand, we can build a positive invariant compact set K_B, as follows, see Figure 10. First, assume $\alpha = 0$, and take any point $B = (-1, y)$ in Σ_-^+ with $y > 0$; after a complete turn around the segment E_C, its orbit arrives again to Σ_-^+ in a point $B_F = (-1, y_F)$ with

$$y_F = P(y) = e^{\pi(\gamma_L+\gamma_R)}y < y.$$

The orbit from B to B_F along the segment $B_F B$ determines a compact positive invariant set. Allowing now α to be positive, the orbit of the same point B will terminate around the turn in a new point $\hat{B}_F = (-1, \hat{y}_F)$; if α is taken small enough, then we can assume that $\hat{y}_F < y$, by using the continuous dependence of solutions with respect to the parameter α. Closing the orbit from B to $\hat{B}_F$ with the segment between these two points on Σ_-, we define the 'big' positive invariant compact set K_B. Obviously, this set K_B can be chosen as small as desired by taking the initial value of y sufficiently small. Furthermore, once fixed the value of α to build the set K_B, the corresponding compact set K_S satisfies $K_S \subset K_B$ for sure, and then we must have a stable limit cycle between the two boundaries of these compact sets. Statemant (b) is done.

The first assertions of statement (c) regarding the birth of the small unstable limit cycle Γ_U, can be obtained by considering the dual case of statement (a), after the replacement $(x, y, \tau) \to (-x, y, -\tau)$.

Finally, the least assertion comes from the fact that as long as the unstable limit cycle Γ_U exists, it defines a compact negative invariant set which must be surround by another stable limit cycle, due to the repulsive character of the point at infinity. □

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