THERE EXIST TRANSITIVE PSVF ON $\mathbb{S}^2$ BUT NOT ROBUSTLY TRANSITIVE.

RODRIGO D. EUZÉBIO, JOABY S. JUCÁ, AND RÉGIS VARÃO

Abstract. It is well known that smooth (or continuous) vector fields do not admit transitive flows on the sphere $\mathbb{S}^2$. Piecewise-smooth vector fields, on the other hand, may present non-trivial recurrences even on $\mathbb{S}^2$. Accordingly, in this paper the presence of transitive flows for piecewise-smooth vector fields (PSVF) on $\mathbb{S}^2$ is proved, see Theorem A. We also prove that every transitive flow occurs in the presence of some particular portions of the phase portrait known as *sliding region* and *escaping region*. More precisely, Theorem B states that under the hypothesis of transitivity, trajectories must interchange between such regions through tangency points. The non-robustness of transitive flows is also proved, see Theorem C. In particular, we prove that there exist no transitive structurally stable flow on $\mathbb{S}^2$. We finish the paper proving Theorem D on robustness of transitive flows for a class of general compact two-dimensional manifolds.

1. Introduction

Piecewise-smooth vector fields (PSVF) are a particular class of dynamical systems for which discontinuities may occur on the phase portrait. Through the literature several authors have dealt with PSVF by assuming distinct conventions to establish how trajectories interact with those discontinuities. A particular approach largely adopted in the study of PSVF was established by A. F. Filippov in [13]. The Filippov convention captures the so-called *sliding motion* in such way that trajectories may reach the discontinuity set in finite time and then slide on that region as well as escape from it. An equivalent approach is considered by Utkin (see [21]), but other conventions can be more restrictive in the treatment of the trajectories, for instance, Barbashin (see [22]) and Broucke et. al. (see [1]). In this paper we adopt the Filippov convention. As a motivation to the study of PSVF on the background of Filippov, we mention that several practical problems can be modeled by such systems, for instance stick-slip process, the antilock braking system (ABS), the relay systems and generally speaking the control theory (see these and other applications of PSVF in [2], [4], [5], [9], [11], [14], [16], [18] and [19]).

An important property of smooth and piecewise-smooth vector fields concerns to topological transitivity. The classical literature on dynamical systems has a well established theory on transitivity for smooth vector fields (an, of course, for diffeomorphims), but such a theory is just premature in the particular context of PSVF. Some preliminary results can be found in [6, 7, 8, 12] and references therein. A PSVF to be topological transitive

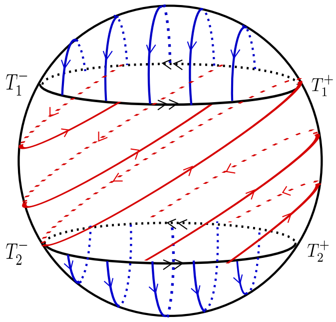

FIGURE 1. Transitive PSVF on $\mathbb{S}^2$

requires that any two open sets can be connected by a Filippov orbit (see the proper definitions on Section 2).

It is a well known result that the two-dimensional sphere does not admits topological transitive continuous vector fields (see [17] and references therein). In this paper, we investigate this and other related issues for PSVF defined on the sphere $\mathbb{S}^2$. In particular, we are able to provide an explicit example of a one-parameter family of topological transitive PSVF on $\mathbb{S}^2$.

Theorem A. *There exist an one-parameter family of transitive piecewise-linear vector fields defined on the sphere $\mathbb{S}^2$.*

One may notice from Figure 1 that the topological transitive vector fields we construct have three zones separated by two circles. A proper question concerns if it is possible to construct a transitive PSVF on the two-dimensional sphere with only two zones. Actually, we are able to partially answer this question. The following result states that for a certain class of PSVF it is not possible when considering two zones separated by a circle.

Proposition 1. *There is no transitive piecewise-linear vector field on $\mathbb{S}^2$ having two zones and separated by a circle.*

Transitive PSVF on $\mathbb{S}^2$ have some inherent features concerning the discontinuities occurring in the phase portrait, we point to the presence of sliding and escaping regions (see Section 2 for precise definitions) connecting to each other through tangency points.

Theorem B. *If Z is a transitive PSVF on $\mathbb{S}^2$ having a finite number of tangency points on Σ, then the following statements hold:*

(a) *the sliding and escaping regions are non-empty sets;*
(b) *every sliding and escaping regions are connected by some trajectory of Z. Moreover there are an uncountable number of trajectories of Z connecting sliding and escaping regions.*

Theorems A and B together illustrate the richness of the trajectories in a PSVF. Nevertheless, every transitive PSVF on $\mathbb{S}^2$ must present a non-trivial recurrence between two type of sets which arise when discontinuities are allowed in the vector fields. On the other hand, such a recurrence is easily broken by small perturbations, that is, transitivity is not a generic property for PSVF as stated in the next result.

Theorem C. *There exist no robustly transitive PSVF on $\mathbb{S}^2$ with finite number of tangency points.*

The next result is a direct consequence of the non-robustness of transitive PSVF.

Corollary 1. *Every transitive Filippov PSVF defined on $\mathbb{S}^2$ with finite number of tangency points is structurally unstable.*

The corollary is proved by noticing that in order to a PSVF fitting the hypotheses of Theorem C be structurally stable, it should also be robustly transitive which is a contradiction to Theorem C. For the sake of completeness we remark that a different proof of Corollary 1 can be achieved using the results on structural stability presented in [1], which applies to Filippov convention due to a suitable definition of topological conjugation considered in that paper. For our purposes only the presented proof will be enough.

We are mainly concerned with $\mathbb{S}^2$ but our techniques can be effortlessly applied to obtain results for compact two-dimensional manifolds.

Theorem D. *Let M^2 be a two-dimensional compact manifold. There exist no robustly transitive PSVF on M^2 having non-empty sliding and escaping regions with finite number of tangency points.*

The next section contains the precise definitions used throughout this paper and the last one contains the proofs of the main results.

2. Preliminaries

Definition 2.1. *A piecewise-smooth vector field is a triple (M, Σ, Z) where*

(i) *M is a suitable manifold;*
(ii) *Σ is formed by a finite union of simple curves $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ splitting M into $n+1$ connected components regions R_i, where $\Sigma_i = \gamma_i^{-1}(0)$ and $\gamma_i : M \to \mathbb{R}$ are smooth functions having 0 as regular values, $i = 1, \dots, n$;*
(iii) *Z is a collection of $n+1$ vector fields of class C^r- defined on M, say $Z = (X_1, \dots, X_{n+1})$, each X_i defined on the closure of R_i.*

We shall denote a PSVF by Z instead of the triple (M, Σ, Z) unless there is some confusion on M or Σ. We call Σ the **switching manifold** and we notice that Z is bi-valuated on Σ. In particular, every component X_i of Z is a vector field defined on whole M which has been restricted to R_i. Because Z is bi-valuated on Σ it is necessary to establish some *rule* describing how trajectories interact to Z, switching to one side of Σ to another or even remaining on Σ. For this purpose, in this paper we adopt the Filippov convention which we describe in this section.

Remark 2.2. We remark that Filippov convention requires connected components of Σ to be disjoint pairwise, simple and smooth. Nevertheless other conventions or including extensions of the Filippov one may suppress some of those assumptions so Definition 2.1 could be slightly adapted.

Let $\Sigma_i = \gamma_i^{-1}(0)$ be the common boundary between the regions R_i and R_j and suppose that $\nabla\gamma_i(p)$ points to the interior of the region R_i for all $p \in \Sigma_i$.

We distinguish three regions on Σ_i satisfying $(X_i.\gamma_i(p))\cdot(X_j.\gamma_i(p)) \neq 0$, where $X_k.\gamma_i(p) = \langle X_k(p), \nabla\gamma_i(p)\rangle$ is the first Lie derivative of γ_i in the direction of vector field X_k at the point p. Such regions characterized in what follows.

Definition 2.3. *Let Σ_i be a connected component of Σ for some $i = 1,\dots,n$. Call X_i and X_j the vector fields separated by Σ_i and let $p \in \Sigma_i$ such that $X_i(p)$ and $X_j(p)$ are transversal to Σ_i at p. Under this assumptions we distinguish three types of regions on Σ_i:*

i) *The* **crossing region** Σ_i^c **of** Σ_i *which is formed by the points $p \in \Sigma_i$ such that $(X_i.\gamma_i(p))\cdot(X_j.\gamma_i(p)) > 0$.*
ii) *The* **escaping region** Σ_i^e **of** Σ_i *which is formed by the points $p \in \Sigma_i$ such that $X_i.\gamma_i(p) > 0$ and $X_j.\gamma_i(p) < 0$.*
iii) *The* **sliding region** Σ_i^s **of** Σ_i *which is formed by the points $p \in \Sigma_i$ such that $X_i.\gamma_i(p) < 0$ and $X_j.\gamma_i(p) > 0$.*

We call $\Sigma^c = \bigcup_{i=1}^{n}\Sigma_i^c$, $\Sigma^s = \bigcup_{i=1}^{n}\Sigma_i^s$ and $\Sigma^e = \bigcup_{i=1}^{n}\Sigma_i^e$ the crossing, sliding and escaping regions of Σ, respectively.

Notice that when $\nabla\gamma_i(p)$ points to the interior of R_j for all $p \in \Sigma_i$ the inequalities in bullets *ii)* and *iii)* of the last definition are interchanged.

Definition 2.4. *The points $p \in \Sigma_i$ such that $X_i.\gamma_i(p) = 0$ (resp. $X_j.\gamma_i(p) = 0$) are called tangency points of X_i (resp. X_j). The collection of the points $p \in \Sigma$ such that p is a tangency point for some vector field X_j, $j = 1,\dots,n+1$ constitute the* **set of tangency points** *of Z denoted by Σ^t.*

Let $p \in \Sigma$ be a tangency point of $Z = (X_1,\dots,X_{n+1})$. We say that Z has a contact of order $n \in \mathbb{N}$ with Σ at p if, for some X_i, $X_i^k.\gamma_i(p) = \langle \nabla X_i^{k-1}.\gamma_i(p), X_i(p)\rangle = 0$ for $k < n$ and $X_i^n.\gamma_i(p) \neq 0$. We classify tangency points according to the following: We say that $p \in \Sigma$ is an *invisible tangency point* if X_i has a contact of even order at p and $X_i^r.\gamma_i(p) < 0$. On the other hand, we say that $p \in \Sigma$ is a *visible tangency point* if either X_i has a contact of even order at p and $X_i^r.\gamma_i(p) > 0$ or X_i has contact of odd order at p.

A particular kind of tangency points are the even contact order points which are tangency points for both X_i and X_j. We refer to those points by *double tangency*. We say that a double tangency p is *elliptic* if p is invisible by X_i and X_j, *hyperbolic* if it is visible by X_i and X_j and *parabolic* if p is visible for X_i and invisible for X_j or otherwise.

In what follows we define the sliding vector field Z^Σ on $\Sigma^{s\cup e}$. If Σ_i is a common boundary of R_i and R_j and $p \in \Sigma_i^s$ then we define $Z^\Sigma(p) = m - p$, with m being the point of the segment joining $p + X_i(p)$ and $p + X_j(p)$ such that $m - p$ is tangent to Σ_i^s (see Figure 2). If $p \in \Sigma_i^e$, then $p \in \Sigma_i^s$ for the vector field $-X$ and we define $Z^\Sigma(p) = -(-X)^\Sigma(p)$. In our pictures we represent the dynamics of Z^Σ by double arrows and we refer to Z^Σ by *Filippov vector field*. The points $p \in \Sigma^{s,e}$ such that $Z^\Sigma(p) = 0$, that is, the equilibrium points of the Filippov vector field, are called *pseudo equilibrium points* of Z.

Remark 2.5. We notice that although the Filippov vector field is defined at sliding and escaping points we can extend it beyond the boundary of $\Sigma^{s,e}$.

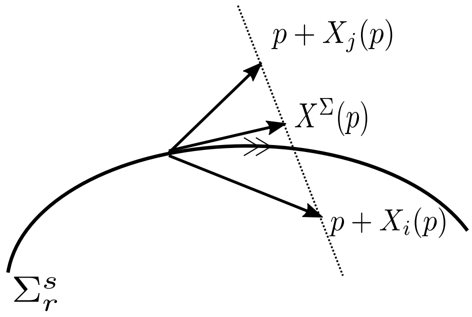

FIGURE 2. Filippov's convention

For instance, if $p \in \Sigma^t$ and

$$\lim_{q \to p} X^{\Sigma}(q) = L \neq 0, \quad q \in \Sigma^{s,e},$$

then we define the extended Filippov vector field at p as $X^{\Sigma}(p) = L$. That will be the case in the proof of Theorem A.

A *global trajectory* $\Gamma_Z(t,p)$ of a piecewise-smooth vector field Z is the trace of a continuous curve obtained by concatenation of trajectories of X_i and/or X_j and/or Z^{Σ} oriented according these vector fields. A *maximal trajectory* $\Gamma_Z(t,p)$ is a global trajectory that cannot be extended by any concatenation of trajectories of X_i, X_j or Z^{Σ}. We also refer to a maximal trajectory $\Gamma_Z(t,p)$ as a *Filippov trajectory (or orbit).* If a Filippov trajectory is singular at $p \in M$ we say that p is an equilibrium point of Z in the sense that $X_i(p) = 0$ for some X_i which is defined on R_i. We say that p is a *real* equilibrium point if p belongs to the closure of R_i, otherwise we say that p is a *virtual* equilibrium point.

We finish this section by introducing two definitions addressing topologically transitive PSVF which are inspired in the classical definitions for smooth vector fields.

Definition 2.6. *A PSVF is topologically transitive if given two arbitrary open sets U and V of M, there exist a Filippov trajectory connecting these sets.*

Definition 2.7. *We say that a Z is robustly topological transitive if Z is topologically transitive and every PSVF sufficiently close to Z is also topologically transitive.*

3. PROOF OF THE MAIN RESULTS

In this section we prove the main results of the paper. We consider the unitary sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ centered at the origin.

3.1. **Proof of Theorem A.** Initially we construct a transitive piecewise-linear vector field on $\mathbb{S}^2$ (see Figure 1). Then we perturb the obtained Filippov vector field to otain the desired one-parametric family of transitive PSVF on $\mathbb{S}^2$.

Effectively, let Σ_1 and Σ_2 be the curves on $\mathbb{S}^2$ given by the intersection of $\mathbb{S}^2$ with the planes $z = 1/2$ and $z = -1/2$, respectively, and consider the linear vector fields

$$X(p) = (z, 0, -x) \quad \text{and} \quad Y(p) = \left(-\frac{1}{2}\left(\sqrt{3}y + z\right), \frac{\sqrt{3}}{2}x, \frac{x}{2}\right)$$

with $p = (x, y, z) \in \mathbb{S}^2$. Let Z be a piecewise-linear vector field with three zones, defined on $\mathbb{S}^2$ by being equal to X on $R_1 = \{(x, y, z) \in \mathbb{S}^2 | \ z \geq 1/2\}$ and $R_3 = \{(x, y, z) \in \mathbb{S}^2 | \ z \leq -1/2\}$ and being equal to Y on $R_2 = \{(x, y, z) \in \mathbb{S}^2 | \ |z| \leq 1/2\}$. Notice that Y is the vector field obtained from $-X$ trough the rotation by the angle $\pi/3$ around of the $x-$axis in the clockwise sense. We claim that Z is topologically transitive on $\mathbb{S}^2$.

In order to prove the claim we easily verify that Z satisfies the following properties:

(i) Each equilibrium point of Z is virtual;

(ii) Z has a pair of double tangency points $T_1^- \neq T_1^+$ with Σ_1, a pair of double tangency points $T_2^- = -T_1^+$ and $T_2^+ = -T_1^-$ with Σ_2, and Z has no more tangency points with Σ besides $T_i^\pm, \ i = 1, 2$;

(iii) the tangency points $T_i^\pm, \ i = 1, 2$ are invisible for X; T_1^+ and T_2^- are visible for Y; T_1^- and T_2^+ are invisible for Y;

(iv) Y has a periodic orbit on $R_2 \cup \Sigma$ that connects T_2^- with T_1^+;

(v) $\Sigma^c = \varnothing$; $\Sigma_1^e = \{(x, y, z) \in \Sigma_1 | \ x < 0\}$, $\Sigma_1^s = \{(x, y, z) \in \Sigma_1 | \ x > 0\}$, $\Sigma_2^e = \{(x, y, z) \in \Sigma_1 | \ x < 0\}$ and $\Sigma_2^s = \{(x, y, z) \in \Sigma_1 | \ x > 0\}$;

(vi) the extended Filippov vector field is well defined on the whole $\Sigma_{1,2}$ and its orientation is in the counter-clockwise sense without pseudo-equilibrium points.

All the items (i) to (vi) can be prove by integrating the linear vector fields X and Y and checking the respective definitions. For example on the item (vi), notice that

$$X_t(p) = (x\cos(t) + z\sin(t), \ y, \ z\cos(t) - x\sin(t))$$

$$\text{and } Y_t(p) = (x_t(p), \ y_t(p), \ z_t(p)), \quad \text{for}$$

$$\begin{cases} x_t(p) = x\cos(t) - 1/2(\sqrt{3}y + z)\sin(t); \\ \\ y_t(p) = \frac{1}{4}(y + 3y\cos(t) + \sqrt{3}(z(-1 + \cos(t)) + 2x\sin(t))); \\ \\ z_t(p) = \frac{1}{4}(3z + \sqrt{3}y(-1 + \cos(t)) + z\cos(t) + 2x\sin(t))), \end{cases}$$

where $p = (x, y, z) \in \mathbb{S}^2$. Then, the extended Filippov vector field on $\Sigma_i, \ i = 1, 2$, can be defined by

$$Z^\Sigma(p) = \frac{YfX - XfY}{Yf - Xf}(p) = \frac{\sqrt{3}}{3}(-y, x, 0), \tag{1}$$

for $p = (x, y, z) \in \Sigma_i^{e,s}, \ i = 1, 2$, and therefore property (vi) is satisfied.

The transitive property of Z follows from the following easily verifiable features of this PSVF (see Figure 1):

- if $|z| \geq 1/2$, $p \neq T_i^\pm, \ i = 1, 2$, then there exist $t > 0$ such that $X_t(p) \in \Sigma_1^s$;
- if $|z| \leq 1/2$, $p \neq T_i^\pm, \ i = 1, 2$, then there exist $t > 0$ such that $Y_t(p) \in \Sigma_2^s$;

- if $p \in \Sigma$ then there exist $t > 0$ such that either $Z_t^{\Sigma}(p) = T_1^+$ for $p \in \Sigma_1$ or $Z_t^{\Sigma}(p) = T_2^-$ for $p \in \Sigma_2$.

Once there exist a periodic orbit for Z on the region R_2 connecting T_1^+ to T_2^- (see the previous item (iv)), then given $p, q \in \mathbb{S}^2$ there exists $t_1,\ t_2 > 0$ and a maximal trajectory $\Gamma_Z(t,p)$ satisfying $\Gamma_Z(t_1,p) = q = \Gamma_Z(-t_2,p)$. Therefore Z is transitive in $\mathbb{S}^2$.

From the previous piecewise-linear transitive vector field we construct a one-parametric family of transitive ones on $\mathbb{S}^2$. Indeed consider the top and the bottom vector fields as previously defined. On the central region, consider the family of vector fields Y_θ with $\pi/6 < \theta \leq \pi/3$, where Y_θ is obtained by rotation of $-X$ by an angle θ in the clockwise sense around the x-axis. Notice that the linear center of Y_θ remains virtual and the new PSVF Z_θ on $\mathbb{S}^2$ is transitive for $\pi/6 < \theta \leq \pi/3$. If $\theta \leq \pi/6$ then the center lies in Σ or it is a real equilibrium of Z and if $\theta > \pi/3$ then there exist no trajectory of R_i visiting R_j for $i, j = 1, 2,\ i \neq j,$. Therefore in this cases the piecewise-linear vector field Z_θ is not transitive on $\mathbb{S}^2$. The verification that Z_θ is transitive for $\pi/6 < \theta \leq \pi/3$ follows a similar argument that we apply to Z and we are done.

3.2. **Proof of Proposition 1.** Let $Z = (X_1, X_2)$ be a linear PSVF defined on $\mathbb{S}^2$ separated by a single circle Σ, that is, we have two regions separated by Σ on the sphere where the linear vector fields X_1 and X_2 are defined. In order to prove the Proposition we have to prove the following assertion for smooth linear vector fields:

If $Z(p) = Ap$ with $A \in M_3(\mathbb{R})$ is a linear vector field defined on $\mathbb{S}^2$, then Z has a pair of antipodes equilibrium points of center type.

The proof of the assertion is direct by noticing that A is necessarily a real skew-symmetric matrix. So $0 \in \mathbb{R}$ is an eigenvalue of A and there are two other conjugated imaginary pure eigenvalues. Therefore the invariant direction corresponding to the eigenvalue 0 generates two antipodal equilibria on $\mathbb{S}^2$, the remaining being periodic on $\mathbb{S}^2$.

Now considering again the piecewise-linear vector field $Z = (X_1, X_2)$. From the above assertion both X_1 and X_2 have antipodal equilibria on $\mathbb{S}^2$. We have two cases:

Case 1: Σ is a great circle. We have three sub-cases to analyse:

- If $X_i,\ i = 1$ or 2 has no equilibrium at Σ, then $Z = (X_1, X_2)$ has a real equilibria of center type.
- If the equilibrium points of X_1 and X_2 belong to Σ and they coincide, then we have two elliptic antipodal tangency points on Σ. In this sub-case we have two situations:

  i) If $\Sigma^c \neq \varnothing$, then $\Sigma = \overline{\Sigma^c}$ and we can delimit an region around the tangency points that are positively or negatively invariant;

  ii) If $\Sigma^c = \varnothing$, then the tangency points separate Σ^s and Σ^e. As these points are also equilibrium points, we have that trajectories that achieve Σ^s do not leave Σ^s.

- If the equilibrium points of X_1 and X_2 belong to Σ and they do not coincide, then we have a sliding segment between two crossing regions and separated by equilibrium points.

In each of the three previous sub-cases we have that Z is not a transitive PSVF on $\mathbb{S}^2$.

Case 2: Σ is not a great circle. Since X_1 and X_2 have antipodal equilibrium points of center type and Σ is a circle on $\mathbb{S}^2$, we have that $Z = (X_1, X_2)$ has at least one real equilibrium point of center type. This make the transitivity impossible and we are done.

3.3. **Proof of Theorem B.**

Proof. We want to prove that Σ^s and Σ^e are non empty sets, hence we start with the following claim:

$$\textit{Claim}\text{: } \Sigma^e \neq \varnothing.$$

The claim is proved by contradiction, hence assume $\Sigma^e = \varnothing$.

The set Σ^s is either empty or nonempty, either case will lead us to a contradiction. Notice that if the sliding set is empty, then we obtain a Filippov system without sliding and escaping region on the sphere. In particular it implies that the only way to a Filippov orbit to experience non-uniqueness of solution is at double tangency points. Hence consider a new vector field denoted by ϕZ, where Z is the Filippov system and ϕ is a positive smooth function which is zero only at the tangency points which are finite by hypothesis. Now we have transformed the Filippov system Z into a vector field with uniqueness of solutions.

This procedure still leaves ϕZ transitive. One way of seeing this is, let γ be a transitive orbit for Z, to be dense it has to go through these tangency points only a finite number of time and at some point it never return to these tangency points. Hence, this forward orbit of γ is dense and coincide to be an orbit of ϕZ (up to a reparametrization) since we only changed the dynamics on the tangency points.

These procedure provides a continuous and transitive vector field on the sphere, which is an absurd. Hence we have eliminated the empty slinding set.

We still have to consider the case where the sliding set is nonempty. Due to transitivity an orbit which enters the sliding region has to leave it. But the exit has a finite number of choices but since we don't have escaping region unless the orbit goes through a tangency it never has uniqueness problem, since we have a finite number of tangencies and transitivity we have that any orbit that leaves a sliding region must return to this sliding region, but it has to be a periodic orbit, and hence not dense. Which is an absurd which finishes the proof of the claim.

The claim implies that any Filippov system as in the hypothesis of the theorem has nonempty escaping region. We now consider a new Filippov system with generating vector fields $\{X_i\}$ replaced by $\{-X_i\}$. This is another way of inverting the orientation of orbits in time and for this new Fillipov system we apply the claim we just proved, hence it has nonempty escaping region. But now the escaping region is the sliding region of the original Filippov system. This proves item a).

We now prove item b). Consider a sliding and an escaping region, respectively, I and J. Let U and V be two open sets close enough to I and J, respectively, in such that any orbit starting on U enters I and any orbit passing through V comes from J. By topological transitivity there exist an orbit that visits U and V. Assume γ is such an orbit with $\gamma(t_1) \in U$ and $\gamma(t_2) \in V$. We assume that $t_1 < t_2$, the other case can be treated by analogous arguments. Since $\gamma(t_1) \in U$ then before reaching V it enters the sliding region and before the time t_2 it touches the escaping region, therefore the sliding and escaping regions I and J are connected.

For the last part of item b) observe that since we can connect sliding and escaping regions consider an orbit that leave some escaping region and enter a sliding region. Notice that small perturbation of this orbit in the escaping region still give an orbit entering the sliding region, hence we can create many different orbits conecting these regions.

□

3.4. **Proof of Theorem C.**

Proof. The proof is done by contradiction. Consider Z a PSVF with a finite number of tangency points and assume that Z is robustly transitive. So there exist U_Z a neighborhood of Z such that every Filippov system in U_Z is transitive.

We first notice that, from Theorem B, Z has sliding and escaping regions, so take $p \in \Sigma^s$ and let X and Y be the adjacent vector fields separated by Σ^s. The positive sliding trajectory of p eventually reaches a tangency point T_A at the boundary of Σ^s, say that X is tangent at T_A. We assume that T_A is not a tangency point for Y and that it is the common boundary between Σ^s and a crossing region (see Figure 3). Otherwise, we perturb Z conveniently in order to obtain the described configuration. We also notice that, if T_A is an invisible tangency point, then it is the ω−limit set of every point nearby Σ^s, but this contradicts the transitivity of Z. Therefore T_A is a visible tangency point of Z and points on Σ^s close to T_A leave Σ through this point. Moreover, this trajectory is unique (see Figure 3).

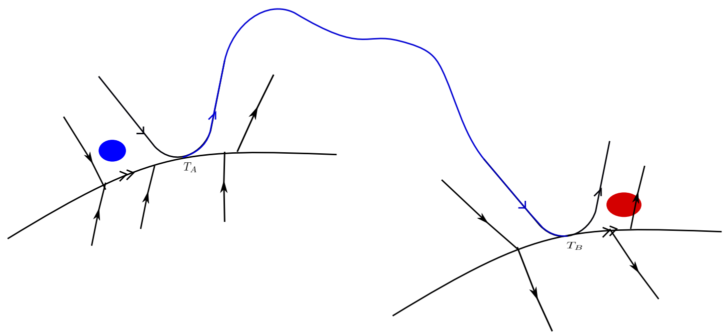

FIGURE 3. Connection between visible tangent points.

Analogously, there exist a visible tangency point T_B located at the common boundary of some escape region Σ^e and a crossing region which repels trajectories of the escaping vector field. In other words, T_B is an entering point (the only one) to the escaping region Σ^e (see Figure 3).

The goal is to use the transitive property to connect T_A to T_B and then perturb $Z = Z_0$ in a suitable way to obtain a topologically transitive PSVF which does not connect tangency points (an absurd from Theorem B).

Effectively, let us call the escaping region associated to T_B by J. Notice that if a point is in a neighborhood of T_B and it is not in a trajectory which enters immediately in J by the tangency T_B, then the time this point takes to enter J is uniformly greater than some fixed time (this is because J has a repelling behavior around it with the exception of the points which enters through T_B the region J itself). Let us call this number by $\alpha > 0$. Since Z_0 is transitive we know by Theorem B that we may connect through a Filippov orbit the tangency points T_A and T_B.

Let $t_0 > 0$ be the time for which an orbit from T_A takes to reach T_B. Let us call this orbit by γ_0, that is, γ_0 is a Filippov orbit of Z_0 such that $\gamma_0(0) = T_A$, $\gamma_0(t_0) = T_B$ and $\gamma_0([0, t_0)) \cap \{T_B\} = \emptyset$.

Let Z_1 be a perturbation of Z_0 which has the following characteristics:

i) $Z_1 \in U_Z$.
ii) Z_1 coincides with Z_0 in V_1^c, where V_1 is an open ball which does not intersect Σ (the switching manifold).
iii) $\gamma_0(t)$ is an orbit of Z_0 and Z_1 as long as $\gamma_0[0, t] \subset V_1^c$.
iv) let γ_1 be an orbit of Z_1 which is a continuation of the orbit γ_0, then there exists a time $t_1 > t_0 + \alpha/2$ such that $\gamma_1([0, t_1)) \cap \{T_B\} = \emptyset$, $\gamma_1(0) = T_A$ and $\gamma_1(t_1) = t_B$.
v) $|Z_0 - Z_1| < 1/2$.

The perturbation is done as follows. We called J the escaping region associated to T_B and let $\mathcal{O}$ be a small neighborhood of J minus $J \cup T_B$ and minus the two connected segments that are inside this neighborhood which connects by a Filippov orbit with the tangency T_B. In other words $\mathcal{O}$ is a small region around J for which the orbits have some sort of repelling behavior. And any point in $\mathcal{O}$ must take more than α to return to the region J.

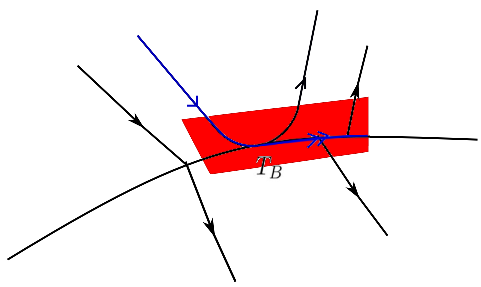

FIGURE 4. Neighborhood $\mathcal{O}$ painted red plus the black trajectories segments contained therein. The blue trajectories do not compose $\mathcal{O}$ and J is the blue continuation of T_B in future time.

Let $\gamma_0(\xi_0)$ be some point very close to T_B and $\xi_0 \in [0, t_0)$. By very close we mean that any ball around $\gamma_0(\xi_0)$ must intersect $\mathcal{O}$, in other words $\gamma_0(\xi_0)$

is so close to T_B that by a perturbation small as we want we can place it inside $\mathcal{O}$. Also, consider a ball V_1 around $\gamma_0(\xi_0)$ small enough such that its closure is inside the closure of $\mathcal{O}$, the closure of V_1 does not intersect Σ and γ_0 enters V_1 in a time greater than $t_0 - \alpha/2$.

We now perturb the vector field Z_0 inside V_1. The perturbation will happen inside some compact set inside V_1. Let us describe this perturbation. On V_1 we define Z_1 to be $Z_0 + W_1$, where W_1 is defined outside V_1 to be zero. Let us define in V_1. We know that $Z_0(\gamma_0(\xi_0))$ is a nonzero vector. Consider v_0 a perpendicular vector to $Z_0(\gamma_0(\xi_0))$. Let V_0 be a smooth vector field around and define $W_1 := \phi V_0$ where ϕ is a smooth bump. We now consider the bump function to be sufficiently small in order that we can guarantee the conditions listed above.

We now proceed in a recursive way. Let Z_n be a perturbation of Z_{n-1} which has the following characteristics:

i) $Z_n \in U_Z$.
ii) Z_n coincides with Z_{n-1} in V_n^c, where V_n is an open ball which does not intersect $\Sigma \cup V_1 \cup \ldots \cup V_{n-1}$.
iii) $\gamma_{n-1}(t)$ is an orbit of Z_{n-1} and Z_n as long as $\gamma_{n-1}[0,t] \subset V_n^c$.
iv) let γ_n be an orbit of Z_n which is a continuation of the orbit γ_{n-1}, then there exists a time $t_n > t_{n-1} + \alpha/2$ such that $\gamma_n([0,t_n)) \cap \{T_B\} = \varnothing$, $\gamma_n(0) = T_A$ and $\gamma_n(t_n) = T_B$.
v) $|Z_n - Z_{n-1}| < 1/2^n$.

Note that Z_n is uniformly converging to $\widetilde{Z}$. Also $\widetilde{Z}$ has never changed Z on Σ. Since $\widetilde{Z}$ is a transitive system it should connect T_A and T_B, but the trajectory of $\widetilde{Z}$ starting at T_A which is the extension of γ_0 never touches T_B. This would give an absurd because we have to conect T_A and T_B, but it turns out that the trajectory of $\widetilde{Z}$ leaving T_A and going to T_B could be a different trajectory instead of the one which is an extension of γ_0 but the other possible way of leaving T_A, hence on the proof above we would have to analyse at the same time both orbits. Hence, we get an absurd. That is, we have a transitive map which does not connect the tangencies T_A and T_B. □

3.5. **Proof of Theorem D.**

Proof. The proof is the same as the proof of Theorem C, we observe that the only place in the proof for which $\mathbb{S}^2$ was really needed was to obtain that for a transitive map on the sphere one has necessarily sliding and escaping region, hence that is the additional hypothesis on the theorem. □

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Instituto de Matemática e Estatística, IME-UFG, Goiânia-GO, Brazil.
Email address: euzebio@ufg.br

Instituto de Matemática e Estatística, IME-UFG, Goiânia-GO, Brazil.
Email address: joabyjuca@discente.ufg.com

Departamento de Matemática, Estatística e Computação Científica, IMECC-UNICAMP, Campinas-SP, Brazil.
Email address: varao@unicamp.br