RICCATI VECTOR FIELDS: AN APPROACH USING MÖBIUS TRANSFORMATIONS

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Abstract. In this paper a class of planar vector fields is characterized and their periodic orbits are studied. The particular class under study is formed by the vector fields whose associated differential system can be written as a Riccati ordinary differential equation. The provided characterization is stated in terms of some homogeneity conditions imposed on the coordinates functions composing the vector fields. The study of periodic orbits is done through Poincaré maps defined around monodromic points. In particular, some properties of the so-called Möbius transformations are considered. In the study both smooth and discontinuous vector fields defined by sectors are studied.

Theorem 1 characterizes the Riccati vector fields and Theorem 2 proves that the Poincaré maps of such vector fields are Möbius transformations. The remaining results of the paper are devoted to study the periodic orbits of Riccati vector fields. More precisely, in Theorem 3 an upper bound of two periodic orbits is stated and we prove this bound is sharp. The polynomial case of degree one and two is considered in Theorems 4 and 5, respectively. In these polynomial cases the existence, upper bound and hiperbolicity of the periodic orbits in Riccati vector fields is stated.

1. Preliminary

1.1. **Introductory remarks.** A central problem in the theory of planar dynamical systems is to determine the existence and distribution of limit cycles as stated by Hilbert in his 16^{th} problem. Hilbert's problem remains open despite of several papers attacking the problem or weakened versions of it. Indeed, a particular case related to Hilbert's problem was proposed by Pugh and revisited by Lins Neto in [1]. Shortly, it consists in relating the problem of finding periodic solutions of planar vector fields in terms of certain non-autonomous ordinary differential equations. The advantage of that approach is the fact there exist a well established theory of ODEs. In particular, several results address the problem of periodic solutions of certain families as Riccati, Abel and Bernoulli equations, for instance. There is an exhaustive list of papers dealing to this subject, we refer to [25] for a nice flavor of the problem.

The Riccati equations are of particular interest because they can have at most two periodic solutions as stated in [1]. The Riccati equations coming from planar vector fields usually are trigonometric ones as studied in [1] and also [19], but other approaches can be found in [6] and [18] and references therein. They are also considered as a simplification of the Abel equations (see [19]), which have been widely studied in the literature, see for instance [20]. Though, in this paper we consider a set of k planar vector fields, $k \geq 1$, whose integral curves are associated to the solutions of k Riccati ODEs, so we call them *Riccati vector fields*. The set of Riccati vector fields considered in this paper is studied inside the theory of discontinuous vector fields since they may present jumps at some portions of the phase portrait. The theory of discontinuous vector fields was started by Andronov at. al in [2] and more recently formalized by Filippov in [15]. Recently, a particular interest in this class of dynamical systems have emerged, one of the reasons being the suitableness to apply the theory in practical problems, see for instance [4], [5], [7], [11] and references therein.

In addition to formalizing the study of Riccati discontinuous vector fields, in this paper we characterize them in terms of some homogeneity conditions. In particular we improve some results presented in [19] for Riccati equations since we assume different homogeneity conditions than those assumed in [19] but also a notion of monodromy which applies for both singular and regular points. As a consequence of the approach, we are able to provide up to two limit cycles while an upper bound of one limit cycle is established in that paper for Riccati equations. A similar characterization for a class of cubic Abel equations can be found in [12]. Ultimately we use the referred characterization and the concept of Möbius transformations to study the limit cycles of Riccati vector fields. As far as the authors know the study of Riccati equations associated to discontinuous vector fields have not been addressed before.

Roughly speaking, in the theory of discontinuous vector fields is considered a partition of the phase portrait in some smaller portions of it being a vector field defined in each portion. The frontier delimiting each region is calling *switching region* and trajectories over this common boundary can be multi-valued, see [15] for the methodology employed in this paper. For this reason, the classical results on dynamical systems may fail. The switching region occurs due to the existence of discontinuities in the vector fields under study. Such discontinuities are usually considered as connected and smooth, most of the cases considered in the literature being straight lines. However, the existence of more general switching regions occurs frequently in applications and are equally important in the theory. In particular, in this paper the shape of the switching region is the so-called *star node* corresponding to a set of straight lines emerging from a common point. This configuration have been considered, for instance, in [8], [9], [23], where the authors search for limit cycles. See also [3] for a general frame and [13] for higher dimensions problems.

We also mention the problem of finding limit cycles for discontinuous linear vector fields which is also considered in this paper along with the quadratic one. When the switching region is a straight line, the maximum number of limit cycles is unknown but every paper published so far indicates that three may be that number, see for instance [14], [16] and [22]. We also refer to the survey presented in [21], which includes some of the several results published on this matter. On the other hand, the linear star node or other configurations on the switching region than not a straight line can increase that upper bound, we mention to [24] and references therein. For instance, a *star node singularity* of a linear system defined on two regions cannot have limit cycles. Nevertheless, by using the approach thought Riccati vector fields and Möbius transformations, in this paper we provide affirmative statements on the existence of limit cycles if one consider three or more distinct regions.

1.2. **Discontinuous vector fields: monodromic points and trajectories.** Let $\{R_i\}_{i=1}^{k-1}$ with $k \in \mathbb{N}$, $k \geq 2$ be a set of half lines emanating from the origin of $\mathbb{R}^2$, each R_i forming an angle θ_i with respect to the positive half *x-axis*. This set of half lines induces a natural partition of $\mathbb{R}^2$ into k connected sectors. We call $\Sigma = \cup_{i=1}^{k-1} R_i$ the switching region and we denote S_i the sectors obtained by the partition $\mathbb{R}^2 \setminus \Sigma$, $i = 1, \ldots, k$. We consider a natural ordering of S_i in such way that R_i is the common boundary between the regions S_{i-1} and S_i. Now on the closure of each S_i we define C^r vector field $X_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $X_i(x, y) = (f_i(x, y), g_i(x, y))$, that is,

$$\begin{gathered} X_1(x,y) = (f_1(x,y), g_1(x,y)) \text{ if } (x,y) \in \overline{S_1}, \\ \vdots \\ X_k(x,y) = (f_k(x,y), g_k(x,y)) \text{ if } (x,y) \in \overline{S_k}. \end{gathered}$$

Notice that every point $p \in \mathbb{R}^2$ is influenced by a unique vector field X_i if $p \notin \Sigma$ or by a pair or vector fields X_i and X_{i+1} otherwise. We call a *discontinuous vector field* the vector field $X(x, y)$ defined on $\mathbb{R}^2$ as

$$X(x,y) = X_i(x,y) \text{ if } (x,y) \in \overline{S_i} \tag{1}$$

and notice that X is bi-valuated at points of Σ. We denote X simply by $X = (X_1, \ldots, X_k)$.

We say that $p \in \mathbb{R}^2$ is a singularity of X if it is a singularity of X_i for some $i \in \{1, \dots, k\}$. In particular, we say that a singularity p of X_i is *real* if $p \in S_i$ or *virtual* if $p \notin \overline{S_i}$. If $p \in \partial S_i$ we say that it is a *boundary singularity*. Finally we say that a point p of X is a *monodromic point* of X if the trajectories of X in $\mathbb{R}^2$ turn around p either in forward and backward time. We notice that a monodromic point of a discontinuous vector field X can be a singularity of X or not. That happens because the discontinuity may induce an orbital regime around a point even if such a point is not a singularity.

We briefly explain what we mean by trajectory of a discontinuous vector field. Let p be an arbitrary point of $\mathbb{R}^2$. Assume that $p \notin \Sigma$, so $p \in S_i$ for some i. The trajectory of X trough p on the interior of S_i is then given by the Existence and Uniqueness Theorem for ODEs. Now assume that eventually such a trajectory reaches R_i or R_{i+1} (say R_{i+1}) transversely in finite positive time and call $q \in R_i$ the intersection point. Notice that q is also defined for the vector field X_{i+1} on the sector S_{i+1} so consider this point as initial condition for X_{i+1}. Depending on the orientation of the trajectory of such vector field some situations can take place at q. For our purposes throughout the paper, we shall assume that the trajectory of X_{i+1} meets q also transversely but now in negative finite time. In this situation we say that q is a *crossing point* for X and the trajectory through p is the local concatenation of the trajectories of X_i and X_{i+1} by q. Now performing similarly with the trajectory of X_{i+1} on the interior of S_{i+1} and suitably concatenating trajectories on ∂S_{i+1} we construct a global trajectory of X through p. We say that such a trajectory is T-*periodic* if p can be concatenated to itself by a finite number of concatenations taking a time T.

We remark that a full classification of points on the switching region is provided by Filippov in [15] but in this paper we only consider crossing points (see Remark 1). For completeness we only mention that reversing orientation the trajectories either of X_i or X_{i+1} one obtain the so-called sliding motion when trajectories slide on the switching region.

1.3. **On the goals of the paper.** The goal of this paper is to characterize a class of planar discontinuous vector fields $X = (X_1, \dots, X_k)$ which are of Riccati type and then to study its periodic orbits surrounding a monodromic point of X. As we will see, X is Riccati type if the differential systems associated to each vector field X_i written in polar coordinates are Riccati ODEs. The study of the periodic orbits is done by defining a Poincaré first return map associated to monodromic points of X and using the so called *Möbius transformations*.

The first main result of the paper provides a characterization of those discontinuous vector fields X which are Riccati, see Theorem 1. The characterization is stated in terms of some homogeneity conditions satisfied by the vector fields. Once classified the discontinuous vector fields of Riccati type, we verify that the Poincaré map we construct, which are formed by a finite composition of transition functions between the switching regions R_i, are actually Möbius transformations. That is the second result of the paper and allow us to provide upper bounds for the number of fixed points of the Poincaré map, see Theorem 2. Provided by the two mentioned theorems, we study the periodic orbits of X. A specific goal of this part is to provide upper bound for the number of limit cycles and also examples of concrete discontinuous vector fields reaching the provided bounds. The answer for this problem is stated for general discontinuous vector fields of Riccati type in Theorem 3. Later, we consider two polynomial situations, where we classify the class of linear and quadratic systems having that Riccati property and we provide specific upper bounds for these two classes, see Theorems 4 and 5, respectively.

Im summary, the results previously described provided an answer to the 16th Hilbert's problem for the class of vector fields of Riccati type. The results also apply to smooth vector fields. Nevertheless, as far as we know, this approach has not been addressed for discontinuous vector fields.

This paper is organized as follows. In Section 2 we introduce the Riccati systems and state Theorem 1 characterizing these systems. In Section 3 we construct the Poincaré map associated to monodromic points, the Möbius transformations and we state Theorem 2 relating these two concepts. An application is also provided in this section. In Section 4 the periodic orbits are considered and we state Theorems 3, 4 and 5. Finally, in Section 5 the main results of the paper are proved and further comments are provided.

2. Riccati systems

2.1. **Setting the problem.** As usual in the theory of dynamical systems a vector field can be associated to a system of differential system. In this paper we make use of such approach to derive a set of ODEs from the discontinuous vector field X defined in (1). There is no special reason to do so but the fact that we will associate planar trajectories of autonomous problems to one-dimensional trajectories of non-autonomous ones by writing down equations into Riccati ones. Therefore, it seems suitable to use write equation (1) as the set of planar differential systems given by

$$\begin{aligned} \dot{x} &= f_i(x, y), \\ \dot{y} &= g_i(x, y), \end{aligned} \quad \text{if } (x, y) \in \overline{S_i}. \tag{2}$$

with $i = 1, \ldots, k$, that we call a *discontinuous differential system* or simply discontinuous system. In addition, throughout the paper we will assume that the origin is a monodromic point and we emphasize that such a point is not necessarily a singularity of system (2) as commented before.

In this paper we classify the discontinuous systems (2) according to some conditions that transform each one of them into Riccati ODEs and then we study the periodic orbits. In order to do this, consider the discontinuous system (2) and assume that is a monodromic point. Changing to polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ then for each $(x, y) \in \overline{S}_i$ we can rewrite system (2) as

$$\begin{aligned} \dot{r} &= \widetilde{f}_i(r, \theta) = \cos\theta f_i(r\cos\theta, r\sin\theta) + \sin\theta g_i(r\cos\theta, r\sin\theta), \\ \dot{\theta} &= \widetilde{g}_i(r, \theta) = r^{-1}[\cos\theta g_i(r\cos\theta, r\sin\theta) - \sin\theta f_i(r\cos\theta, r\sin\theta)], \end{aligned} \tag{3}$$

for $i = 1, \ldots, k$. Now assume that $\widetilde{g}_i$ does not vanish on $\overline{R}_i$ and the quotient $\dot{r}/\dot{\theta}$ is a polynomial of degree n in the variable r. We thus can write systems (3) into k one-dimensional non-autonomous differential equations of the form

$$\frac{dr}{d\theta} = a^i_{n-1}(\theta) r^{n-1} + \ldots + a^i_1(\theta) r + a^i_0(\theta), \tag{4}$$

where each $a^i_j(\theta)$ are quotients of continuous functions in terms of $\cos\theta$ and $\sin\theta$. We say the discontinuous ordinary differential equation (4) has a *closed solution* γ if it is defined in the interval $[0, 2\pi]$ and $\gamma(0) = \gamma(2\pi)$. As we will see, closed solutions for equation (4) corresponds to periodic orbits of system (2).

Definition 1. *We say that system (2) is a Riccati discontinuous system (or discontinuous system of Riccati type) if for every $i = 1, \ldots, k$ the ordinary differential equation (4) writes*

$$\frac{dr}{d\theta} = a^i_2(\theta) r^2 + a^i_1(\theta) r + a^i_0(\theta). \tag{5}$$

We now present the first main result of the paper. For this we consider the following general hypotheses on system (2) for $i = 1, \ldots, k$.

- **H.1:** $xg_i(x, y) - yf_i(x, y) := \varphi_i(x, y) \neq 0$, where $\varphi_i(\rho x, \rho y) = \rho^n \varphi_i(x, y)$, for all $\rho > 0$ with $n \in \mathbb{N}$;
- **H.2:** $xf_i(x, y) + yg_i(x, y) := \psi_i(x, y)$, where $\psi_i(\rho x, \rho y) = \rho^{n+1} A_i(x, y) + \rho^n B_i(x, y) + \rho^{n-1} C_i(x, y)$, for all $\rho > 0$, where A_i, B_i and C_i are continuous functions.

Now we have the following result.

Theorem 1. *System* (2) *is a Riccati discontinuous system if, and only if, it satisfies hypotheses* $H.1$ *and* $H.2$ *for every* $i = 1, \dots, k$.

In the coming sections we introduce the other results of the paper which depend on the concepts of the Poincaré first return maps and Möbius transformations as we state next.

3. Poincaré maps and Möbius transformations

3.1. **The Poincaré map associated to monodromic points.** Let X be a discontinuous vector field and Σ the associated switching region. In this section we define a first return map for a $T-$periodic trajectory Γ around the origin which is the only monodromic point of X since we are assuming only crossing points on Σ. Hence because the origin is monodromic the $T-$periodic trajectory Γ intersects Σ in $k-1$ points $p_1, \dots, p_{k-1}$ being $p_i \in R_i$, $i = 1, \dots, k-1$. We call $\varphi_t(p) = \varphi(t, p)$ the trajectory of p at t associated to system (2) and we set $\varphi(0, p_1) = p_1 \in R_1$ so

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \,|\, (x, y) = \varphi_t(p_1),\ 0 \leq t \leq T\}.$$

More precisely, we obtain real values $0 = t_0 < \dots < t_{k-1} < \infty$ such that $t_0 + \dots + t_{k-1} = T$ and

$$\varphi(t_1, p_1) = p_2 \in R_2, \dots, \varphi(t_{k-2}, p_{k-2}) = p_{k-1} \in R_{k-1},\ \varphi(t_{k-1}, p_{k-1}) = p_1 \in R_1.$$

Let $V_{p_1} \subset R_1$ be an arbitrarily small neighborhood of p_1 in R_1. For each $z_1 \in V_{p_1}$ we define the transition map

$$\begin{aligned} P_1 \quad : \quad & V_{p_1} \longrightarrow R_2 \\ & z_1 \mapsto P_1(z_1) = \varphi_{t(z_1)}(z_1) \end{aligned}$$

and observe that $P_1(z_1)$ is arbitrarily close to p_2 when z_1 is arbitrarily close to p_1. Using the same construction we can define $k-1$ more transition functions $P_2, \dots, P_k$ in such way that $P = P_k \circ \dots \circ P_1 : R_1 \longrightarrow R_1$ is the first return map on R_1. Of course, $P(p_1) = p_1$ corresponds to the $T-$periodic orbit Γ. Clearly, P is a diffeomorphism being $P^{-1}(z_1) = \varphi_{-t(z_1)}(z_1)$.

In this paper is very useful to consider the first return map in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$. Nevertheless, there is no loss of generality in assuming that the origin has been translated to $p_1 \in R_1$ in such way that $p_1 = (0, 0)$, $V_{p_1} \simeq (-\varepsilon, \varepsilon) \subset \mathbb{R}$ with $\varepsilon > 0$ sufficiently small and $P : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$. Therefore, in polar coordinates, we can assume that R_1 is the ray $\theta = \theta_0 = 0$ through the origin so that it is transversal to the $T-$ periodic orbit Γ at $(\widetilde{r}_0, 0)$. More generally, the trajectory of any point $(r, \theta) = (r_0, 0) \in R_1$ at $t = 0$ intersects again the ray $\theta = 0$ at $t = 2\pi$. Calling Π the expression of P at $(r, \theta) = (r_0, 2\pi)$ and Π_i the transition functions P_i we then obtain the first return map in polar coordinates which is a r_0-parametric function satisfying $\Pi(\widetilde{r}_0) = \widetilde{r}_0$ being $\Pi = \Pi_k \circ \dots \circ \Pi_1$.

It is also useful to define the so-called *displacement function* $d = d(r_0)$ which is the defined by $d(r_0) = \Pi(r_0) - r_0$. Clearly, a fixed point $\widetilde{r}_0$ for the first return map correspond to a zero of the displacement function, and in both situations a periodic orbit Γ of system (2) takes place. As usual, we say that Γ is *hyperbolic* if $d'(r_0) \neq 0$ and in this case Γ is *stable* if $d'(r_0) < 0$ and *unstable* if $d'(r_0) > 0$. In the non-hyperbolic scenario the stability must be defined in terms of neighborhoods of γ but it will not be considered in this paper.

Remark 1. *As mentioned before in this paper we only consider crossing points on* Σ. *It means trajectories reaching the boundary of each sector just switch from one sector to another in a continuous (not necessarily smooth) way, see Figure 1. However, the strong hypothesis behind the results concerns the existence of a first*

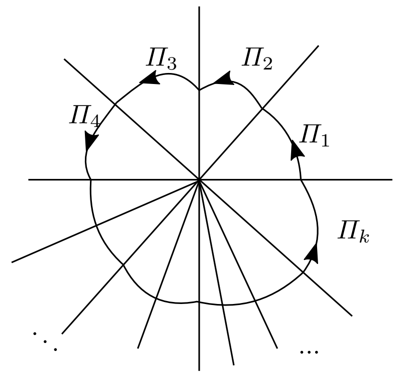

FIGURE 1. Scheme of the Poincaré map Π defined by sectors.

return map which can be defined even in the presence non-transversal points or sliding, that is, assuming that trajectories slide along Σ. It could be done by shrinking the returning region suitably. In particular, there is no loss of generality in the assumption that we impose on existence of only crossing points.

3.2. **Möbius transformations.** We have defined above the Poincaré map which is a composition of transitions functions. Now, we will show that Poincaré maps associated to Riccati discontinuous systems are in fact a particular kind of transformation, namely, the Möbius' ones. In this paper we define and present two facts about Möbius transformations. For more information about this matter, one can consult for instance [10].

Definition 2. *An transformation $T : \mathbb{C} \to \mathbb{C}$ is of Möbius type if T is a rational function of the form*

$$T(z) = \frac{az + b}{cz + d}, \tag{6}$$

where the coefficients satisfy $ad - bc \neq 0$.

The main result on Möbius transformation we use in this paper concerns the number of its fixed points and the invariance of the Möbius property under compositions. We have included this results next for the sake of completeness, but they can also be found in [10]. The proof is elementary.

Proposition 1. *Every Möbius transformation has at most two fixed points. Moreover, the composition of two Möbius transformations is again a Möbius transformation.*

In what follows we present the second main result of the paper.

Theorem 2. *Consider system (2) and let X be its associated vector field. If (2) is a Riccati discontinuous systems and the origin is a monodromic point of X, then the transitions functions $\Pi_i : V_{p_i} \longrightarrow N_{i+1}$ from $p_i \in N_i$ to $\Pi_i(p_i) \in N_{i+1}$ are Möbius transformations for every $i = 1, \dots, k$.*

Theorem 2 will be the main result for proving the results on periodic orbits in the next section. These statements can be verified in the next application.

3.3. **Application to systems having constant angular speed.** Next we present an application of a Riccati discontinuous system composed by subsystems having constant angular speed. This problem is largely studied in the literature because of its importance in the *center-focus problem*. We refer to [17] and references therein to a brief view in this subject. This kind of systems are also called *rigid* or *uniformly isochronous* systems. Indeed, if the origin is non-degenerated a rigid system can be written as

$$\begin{aligned}\dot{x} &= \alpha y + x\, F(x,y),\\ \dot{y} &= -\alpha x + y\, F(x,y),\end{aligned}$$

where F is an arbitrary function. For instance, in [17] the authors deal with a polynomial quadratic function F and they prove that some classes have at most one periodic orbit. In the above example, we can reach the same bound of one limit cycle but assuming that F is a polynomial linear function. This highlight that discontinuous systems present a more rich dynamics than smooth ones.

The aim of presenting this application here is to provide a distinguished class of systems fitting our hypotheses but also to exemplify the construction of the Poincaré map, the role of hypotheses **H.1** and **H.2** and of the Möbius transformations in the context of obtaining periodic orbits.

Consider the piecewise differential system $X = (X^+, X^-)$ separated by one straight line with

$$X^+ : \begin{cases} \dot{x} = P_1(x,y) = b_1 y + x(a_1 + c_1 x + d_1 y), \\ \dot{y} = Q_1(x,y) = -b_1 x + y(a_1 + c_1 x + d_1 y), \end{cases} \quad \text{if} \quad y \geq 0,$$

and

$$X^- : \begin{cases} \dot{x} = P_2(x,y) = b_2 y + x(a_2 + c_2 x + d_2 y), \\ \dot{y} = Q_2(x,y) = -b_2 x + y(a_2 + c_2 x + d_2 y), \end{cases} \quad \text{if} \quad y \leq 0.$$

We suppose that $b_1 b_2 > 0$, so locally around the equilibrium point located at the origin the trajectories of both X^+ and X^- rotate around it in the same sense. Therefore, the origin is a monodromic point for X which is a discontinuous vector field having $\Sigma = \{y = 0\}$ as switching manifold. One can notice that $X^+(x,0)$ and $X^-(x,0)$ points to the same side of Σ because $b_1 b_2 > 0$, so every point on that region is formed by crossing points. We shall prove that there is at most one limit cycle around origin.

Observe that the coordinates P_j and Q_j of the vector fields X^+ and X^- satisfy for $j = 1,2$ the following conditions.

$$\begin{aligned} xQ_j(x,y) - yP_j(x,y) &= \varphi_j(x,y) = -b_j\left(x^2 + y^2\right), \\ xP_j(x,y) + yQ_j(x,y) &= \psi_j(x,y) = a_j x^2 + a_j y^2 + c_j x^3 + c_j x y^2 + d_j x^2 y + d_j y^3. \end{aligned}$$

Moreover, φ and ψ satisfy

$$\begin{aligned} \varphi_j(\rho x, \rho y) &= -\rho^2 b_j\left(x^2 + y^2\right) = \rho^2 \varphi_j(x,y), \\ \psi_j(\rho x, \rho y) &= \rho^3\left(c_j x^3 + c_j x y^2 + d_j x^2 y + d_j y^3\right) + \rho^2\left(a_j x^2 + a_j y^2\right), \end{aligned}$$

that is, X fulfills the hypotheses **H.1** and **H.2**. Now, we construct the Poincaré map and we show that it is a Möbius transformation. Indeed, changing to polar coordinates $(x,y) \longrightarrow (r,\theta)$ with $x = r\cos\theta, y = r\sin\theta$ we get

$$(\dot{r}, \dot{\theta}) = \begin{cases} (r^2(c_1\cos\theta + d_1\sin\theta) + a_1 r, -b_1) & \text{if} \quad 0 \leq \theta \leq \pi, \\ (r^2(c_2\cos\theta + d_2\sin\theta) + a_2 r, -b_2) & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

Setting θ as the new independent variable we obtain

$$r' = \begin{cases} -\dfrac{c_1\cos\theta + d_1 sin\theta}{b_1} r^2 - \dfrac{a_1}{b_1} r & \text{if} \quad 0 \leq \theta \leq \pi, \\ -\dfrac{c_2\cos\theta + d_2 sin\theta}{b_2} r^2 - \dfrac{a_2}{b_2} r & \text{if} \quad \pi \leq \theta \leq 2\pi, \end{cases}$$

therefore X is a Riccati discontinuous system according to Definition 1. The prime is the last equation denotes the derivative respect to θ. Moreover, given r_0 and r_1, the Poincaré map is given by $\Pi(r_0) = \Pi_2(\Pi_1(r_0))$

with

$$\Pi_1(r_0) = \frac{\left(a_1^2 + b_1^2\right) r_0}{\left(e^{\frac{\pi a_1}{b_1}} + 1\right)(a_1 c_1 + b_1 d_1) r_0 + (a_1^2 + b_1^2)\, e^{\frac{\pi a_1}{b_1}}}, \text{if} \quad 0 \le \theta \le \pi,$$

and

$$\Pi_2(r_1) = \frac{\left(a_2^2 + b_2^2\right) r_1}{-\left(e^{\frac{\pi a_2}{b_2}} + 1\right)(a_2 c_2 + b_2 d_2) r_1 + (a_2^2 + b_2^2)\, e^{\frac{\pi a_2}{b_2}}}, \text{if} \quad \pi \le \theta \le 2\pi,$$

see Figure 2.

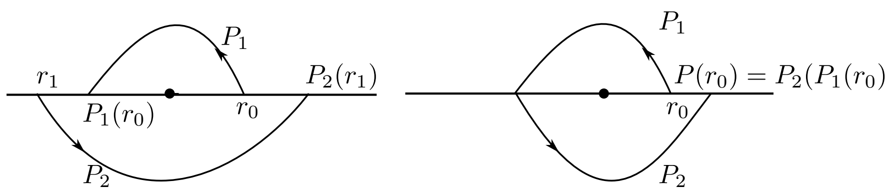

Figure 2. First Return.

Therefore, according to Definition 6 both Π_1 and Π_2 are Möbius transformations so by Proposition 1 $\Pi(r_0) = \Pi_2 \circ \Pi_1$ is also Möbius . More precisely, we have

$$\Pi(r_0) = \frac{(a_1^2 + b_1^2)(a_2^2 + b_2^2)\, r_0}{(a_1^2 + b_1^2)(a_2^2 + b_2^2) e^{\frac{a_1}{b_1}\pi + \frac{a_2}{b_2}\pi} + \alpha\, r_0},$$

with $\alpha = \alpha(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2)$ is given by

$$\begin{aligned} \alpha \quad = \quad & a_1(a_2^2 + b_2^2) c_1 e^{\frac{a_2}{b_2}\pi}\left(1 + e^{\frac{a_1}{b_1}}\right) - a_1^2(a_2 c_2 + b_2 d_2)\left(1 + e^{\frac{a_2}{b_2}}\right) + b_1(-b_1(a_2 c_2 + \\ & + b_2 d_2) + (a_2 b_1 c_2 + a_2^2 d_1 + b_2(b_2 d_1 - b_1 d_2)) e^{\frac{a_2}{b_2}\pi} + d_1(a_2^2 + b_2^2) e^{\frac{a_1}{b_1}\pi + \frac{a_2}{b_2}\pi}. \end{aligned}$$

which is Möbius so using again Proposition 1 it has at most two fixed points. Clearly $r_0 = 0$ is a solution that does not correspond to a periodic orbit. The other fixed point is given by

$$\widetilde{r_0} = \frac{1}{\alpha}(a_1^2 + b_1^2)(a_2^2 + b_2^2)\left(1 - e^{\frac{a_1}{b_1}\pi + \frac{a_2}{b_2}\pi}\right)$$

provided that $\widetilde{r_0} > 0$. Therefore system X has one periodic orbit and not other can take place.

4. Periodic orbits of Riccati discontinuous systems

This section establishes some results about the upper bounds for the number of discontinuous systems of Riccati type. The problem of finding periodic orbits is widely studied both in smooth and discontinuous scenarios and it is related to Hilbert 16th problem. The main aspects of periodic orbits concern the existence, upper bounds and realization of them, hiperbolicity, stability and distribution of them and in this paper we provide most of these properties the class of Riccati discontinuous systems.

The following theorem is the third main result of the paper.

Theorem 3. *Consider system* (2) *and let X be its associated vector field. If system* (2) *is a Riccati discontinuous system and the origin is a monodromic point of X, then it has at most two limit cycles surrounding the origin and this bound is reached.*

The three previous results of the paper concern general systems. In what follows we consider the particular case of polynomial systems of degree one and two.

Theorem 4. *Assume that the origin is a monodromic point for system (2) with* $f_i(x,y) = p^i_{00} + p^i_{10}x + p^i_{01}y$ *and* $g_i(x,y) = q^i_{00} + q^i_{10}x + q^i_{01}y$, $j = 1,\dots,k$, *being* k *the number of sectors defining system (2). Thus, system (2) is of Riccati type if, and only if, for every* $j \in \{1,\dots,k\}$ *either* $p^j_{00} = q^j_{00} = 0$ *or* $p^j_{01} = q^j_{10} = 0$ *and* $p^j_{10} = q^j_{01}$. *That is, there exists only two different families of Riccati systems* $\mathcal{L}^j_1$ *and* $\mathcal{L}^j_2$ *defined in the different sectors* S_j *and they write*

$$\mathcal{L}^j_1 : \begin{cases} \dot{x} = f_j(x,y) = p^j_{10}x + p^j_{01}y, \\ \dot{y} = g_j(x,y) = q^j_{10}x + q^j_{01}y, \end{cases} \quad \text{and} \quad \mathcal{L}^j_2 : \begin{cases} \dot{x} = f_j(x,y) = p^j_{10}x + p^j_{00} \\ \dot{y} = g_j(x,y) = p^j_{10}y + q^j_{00}. \end{cases}$$

Moreover, the following statements hold.

(i) *if system (2) is composed by at least one family* $\mathcal{L}^j_1$, *for some* j, *then it has at most one limit cycle;*

(ii) *if system (2) has a subsystem of family* $\mathcal{L}^j_2$ *then* $k \geq 3$ *and the equilibrium point of every family* $\mathcal{L}^j_2$ *is virtual;*

(iii) *there are systems of the form (2) having one limit cycle. More precisely, we can exhibit a value* $\widetilde{k}$, *a switching manifold* $\widetilde{\Sigma}$ *and a set of families* $\{\widetilde{\mathcal{L}}^1_\ell, \dots, \widetilde{\mathcal{L}}^{\widetilde{k}}_\ell\}$, $\ell \in \{1,2\}$, *such that one limit takes place in the provided Riccati discontinuous system.*

We notice that if a linear system is of Riccati type and has an equilibrium point, then it is either a star node in the second case or a general homogeneous linear system in the first one.

Next result extends the previous one for quadratic systems. For this class, however, we are not able to provide a system reaching the upper bound we state.

Theorem 5. *Assume that the origin is a monodromic point for system (2) with* $f_i(x,y) = \sum^2_{0\leq r+s\leq 2} p^i_{rs}x^ry^s$ *and* $g_i(x,y) = \sum^2_{0\leq r+s\leq 2} q^i_{rs}x^ry^s$, $j = 1,\dots,k$, *being* k *the number of sectors defining system (2). Thus, system (2) is of Riccati type if, and only if, for every* $j \in \{1,\dots,k\}$ *either* $q^j_{20} = p^j_{00} = p^j_{02} = q^j_{00} = 0, q^j_{11} = p^j_{20}, q^j_{02} = p^j_{11}$ *or* $p^j_{00} = p^j_{01} = q^j_{10} = q^j_{00} = 0, q^j_{01} = p^j_{10}$. *That is, there exists only two different families of Riccati systems* $\mathcal{Q}^j_1$ *and* $\mathcal{Q}^j_2$ *defined in the different sectors* S_j *and they write*

$$\mathcal{Q}^j_1 : \begin{cases} \dot{x} = f_j(x,y) = p^j_{10}x + p^j_{01}y + p^j_{20}x^2 + p^i_{11}xy, \\ \dot{y} = g_j(x,y) = q^j_{10}x + q^j_{01}y + p^j_{20}xy + p^j_{11}y^2, \end{cases} \quad \text{and} \quad \mathcal{Q}^j_2 : \begin{cases} \dot{x} = f_j(x,y) = p^j_{10}x + p^j_{20}x^2 + p^j_{11}xy + p^j_{02}y^2, \\ \dot{y} = g_j(x,y) = p^j_{10}y + q^j_{20}x^2 + q^j_{11}xy + q^j_{02}y^2. \end{cases}$$

Moreover, the following statements hold.

(i) *system (2) has at most two limit cycles;*

(ii) *there are systems of the form (2) having one limit cycle. More precisely, we can exhibit a value* $\widetilde{k}$, *a switching manifold* $\widetilde{\Sigma}$ *and a set of families* $\{\widetilde{\mathcal{Q}}^1_\ell, \dots, \widetilde{\mathcal{Q}}^{\widetilde{k}}_\ell\}$, $\ell \in \{1,2\}$, *such that one limit takes place in the provided Riccati discontinuous system.*

Remark 2. *We highlight that the results presented in the paper also holds for smooth systems which is the case* $k = 1$. *Some differences however emerge as in bullet (c) of Theorem 4. Also, notice that in both Theorems 4 and 5 the monodromic points may be a regular point, not necessarily an equilibrium point. However, setting* $k = 1$ *by Tubular Flow Theorem one only can observe monodromy at equilibrium points. Other number of sectors with* $k \leq 2$ *should be studied separately as we done in bullet (c) of Theorem 4.*

We now prove the main results of the paper.

5. Proof of the main results and further discussions

In this section we prove the main results of the paper. In order to prove Theorem 1, we first state and prove the following result.

Lemma 1. *Consider the planar differential system*

$$\begin{aligned} \dot{x} &= f(x,y),\\ \dot{y} &= g(x,y), \end{aligned} \tag{7}$$

and assume that in polar coordinates it can be written into the form

$$\begin{aligned} \dot{r} &= A(\theta)r^n + B(\theta)r^{n-1} + C(\theta)r^{n-2},\\ \dot{\theta} &= D(\theta)r^{n-2}, \end{aligned}$$

where A, B, C and D are differential functions depending on f and g. Then system (7) satisfies hypotheses **H.1** *and* **H.2**.

Proof. Consider the functions

$$\varphi(x,y) = \left(x^2+y^2\right)^{\frac{n}{2}} D\left(\tan^{-1}\left(\frac{y}{x}\right)\right),$$

$$\psi(x,y) = \left(x^2+y^2\right)^{\frac{n+1}{2}} A\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \left(x^2+y^2\right)^{\frac{n}{2}} B\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \left(x^2+y^2\right)^{\frac{n-1}{2}} C\left(\tan^{-1}\left(\frac{y}{x}\right)\right),$$

which clearly satisfy the homogeneity conditions given in **H.1** and **H.2**. Moreover, by hypothesis in polar coordinates they satisfy

$$\begin{aligned} xg(x,y) - yf(x,y) &= r^2\dot{\theta} = r^2 D(\theta) r^{n-2} = r^n D(\theta),\\ xf(x,y) + yg(x,y) &= r\dot{r} = A(\theta)r^{n+1} + B(\theta)r^n + C(\theta)r^{n-1}. \end{aligned}$$

Hence, returning to the original variables we get that $r^n D(\theta) = \varphi(x,y)$ and $A(\theta)r^{n+1} + B(\theta)r^n + C(\theta)r^{n-1} = \psi(x,y)$ so φ and ψ are the desired functions, that is, system (7) satisfies hypotheses **H.1** and **H.2**. □

Proof of Theorem 1. Consider system (2) which in polar coordinates write, for $i = 1,\dots,k$,

$$\begin{aligned} \dot{r} &= \frac{xf_i(x,y) + yg_i(x,y)}{r},\\ \dot{\theta} &= \frac{xg_i(x,y) - yf_i(x,y)}{r^2}, \end{aligned}$$

where $x = r\cos\theta, y = r\sin\theta$. First, suppose that hypothesis **H.1** and **H.2** are fulfilled. Thus, we obtain that $xf_i(x,y) + yg_i(x,y) = \psi_i(x,y)$, $xg_i(x,y) - yf_i(x,y) = \varphi_i(x,y)$. Moreover,

$$\begin{aligned} \dot{r} &= \frac{\psi_i(x,y)}{r},\\ \dot{\theta} &= \frac{\varphi_i(x,y)}{r^2}. \end{aligned}$$

Also, replacing $x = r\cos\theta, y = r\sin\theta$, we get

$$\begin{aligned} \dot{r} &= \frac{\psi_i(r\cos\theta, r\sin\theta)}{r} = \frac{r^{n+1}A_i(\cos\theta,\sin\theta) + r^n B_i(\cos\theta,\sin\theta) + r^{n-1}C_i(\cos\theta,\sin\theta)}{r},\\ \dot{\theta} &= \frac{\varphi_i(r\cos\theta, r\sin\theta)}{r^2} = \frac{r^n\varphi_i(\cos\theta,\sin\theta)}{r^2}, \end{aligned}$$

and then

$$\frac{dr}{d\theta} = \frac{A_i(\cos\theta,\sin\theta)}{\varphi_i(\cos\theta,\sin\theta)}r^2 + \frac{B_i(\cos\theta,\sin\theta)}{\varphi_i(\cos\theta,\sin\theta)}r + \frac{C_i(\cos\theta,\sin\theta)}{\varphi_i(\cos\theta,\sin\theta)}.$$

Therefore, system (2) is a Riccati discontinuous system.

Reciprocally, suppose that the planar discontinuous system is of Riccati type. Then, for each $i = 1, \ldots, k$, the systems

$$\begin{aligned} \dot{x} &= f_i(x, y), \\ \dot{y} &= g_i(x, y), \end{aligned}$$

in polar coordinates write

$$\begin{aligned} \dot{r} &= F_i(r, \theta), \\ \dot{\theta} &= G_i(r, \theta), \end{aligned} \tag{8}$$

in such way that, taking θ as the new independent variable, it writes

$$\frac{dr}{d\theta} = \frac{F_i(r,\theta)}{G_i(r,\theta)} = a_2^i(\theta)r^2 + a_1^i(\theta)r + a_0^i(\theta).$$

Note that system (8) is equivalent to

$$\begin{aligned} \dot{r} &= \frac{F_i(r,\theta)}{G_i(r,\theta)} = a_2^i(\theta)r^2 + a_1^i(\theta)r + a_0^i(\theta), \\ \dot{\theta} &= 1. \end{aligned}$$

so, from Lemma 1, system (2) satisfies the hypotheses **H.1** and **H.2**. □

Proof of Theorem 2. From Theorem 1, system (2) is of Riccati type so for every $i = 1, \ldots, k$, it can be written in polar coordinates is an ordinary differential equation of Riccati type. Thus, on region R_i it writes

$$\frac{dr}{d\theta} = a_2^i(\theta)r(\theta)^2 + a_1^i(\theta)r(\theta) + a_0^i(\theta), \tag{9}$$

with initial condition $r(\theta_i) = r_0^i$. Let $r_p^i(\theta)$ be a particular solution of equation (9) on the region R_i and consider, for every $i = 1, \ldots, k$, the following change of variables

$$u_i(\theta) = \frac{1}{r(\theta) - r_p^i(\theta)}.$$

so we obtain

$$\frac{du_i}{d\theta} = -u_i(\theta)(a_1^i(\theta) + 2a_2^i(\theta)r_p^i(\theta)) - a_2^i(\theta). \tag{10}$$

Solving equation (10) with $u_i(\theta_i) = u_0^i$, we obtain

$$u_i(\theta) = e^{\displaystyle\int_{\theta_i}^{\theta} (-a_1^i(\tau) - 2a_2^i(\tau)r_p^i(\tau))\, d\tau} \left(\int_{\theta_i}^{\theta} a_2^i(s) \left(-e^{-\int_{\theta_i}^{s} (-a_1^i(\tau) - 2a_2^i(\tau)r_p^i(\tau))\, d\tau} \right) ds + u_0^i \right).$$

Going back to the variable r, we get

$$r(\theta, r_0^i) = \frac{m_i(\theta)r_0^i + n_i(\theta)}{M_i(\theta)r_0^i + N_i(\theta)},$$

where

$$
\begin{aligned}
M_i(\theta) &= \int_{\theta_i}^{\theta} a_2^i(s)\left(-e^{-\int_{\theta_i}^{s}(-a_1^i(\tau)-2a_2^i(\tau)r_p^i(\tau))\,d\tau}\right) ds,\\
N_i(\theta) &= 1-r_p^i(\theta_i)\left(\int_{\theta_i}^{\theta} a_2^i(s)\left(-e^{-\int_{\theta_i}^{s}(-a_1^i(\tau)-2a_0^i(\tau)r_p^i(\tau))\,d\tau}\right) ds\right),\\
m_i(\theta) &= e^{-\int_{\theta_i}^{\theta}(-a_1^i(\tau)-2a_2^i(\tau)r_p^i(\tau))\,d\tau} + r_p^i(\theta)\left(\int_{\theta_i}^{\theta} a_2^i(s)\left(-e^{-\int_{\theta_i}^{s}(-a_1^i(\tau)-2a_2^i(\tau)r_p^i(\tau))\,d\tau}\right) ds\right)\\
n_i(\theta) &= -r_p^i(\theta_i)e^{-\int_{\theta_i}^{\theta}(-a_1^i(\tau)-2a_2^i(\tau)r_p^i(\tau))d\tau} +\\
&\quad + r_p^i(\theta)r_p^i(\theta_i)\left(-\int_{\theta_i}^{\theta} a_2^i(s)\left(-e^{-\int_{\theta_i}^{s}(-a_1^i(\tau)-2a_2^i(\tau)r_p^i(\tau))d\tau}\right) ds\right) + r_p^i(\theta).
\end{aligned}
$$

Therefore, since the origin is a monodromic point, the transition map in the region R_i is well defined and it writes

$$
r(\theta_{i+1}, r_0^i) = \Pi_i(r_0^i) = \frac{m_i(\theta_{i+1})r_0^i + n_i(\theta_{i+1})}{M_i(\theta_{i+1})r_0^i + N_i(\theta_{i+1})},
$$

that is, Π_i is a Möbius transformations for every $i = 1, \ldots, k$. □

Proof of Theorem 3. First we prove the upper bound for the number of limit cycles. Indeed, from Theorem 1 on each region we obtain a Riccati discontinuous system. Moreover, from Theorem 2 the transitions maps Π_i, $1 \leq i \leq k$ are Möbius transformations and they are well defined because the origin is a monodromic point. Therefore, the Poincaré map, which is a composition of k transition maps, writes

$$
\Pi = \Pi_{\widetilde{k}} \circ \Pi_{\widetilde{k}-1} \circ \Pi_{\widetilde{k}-2} \circ \ldots \circ \Pi_2 \circ \Pi_1,
$$

and it is also a Möbius transformations from Proposition 1. Hence,, the Poincaré map Π can be written into the form

$$
\Pi(r) = \frac{ar+b}{cr+d},
$$

with $a \neq 0$, where the coefficients a, b, c and d depend on every m_i, M_i, n_i and N_i for each $i = 1, \ldots, k$. Therefore, using again Proposition 1, $\Pi(r)$ has at most two fixed points, that is, system (2) has at most two limit cycles because at most two periodic orbits are isolated.

Now we prove that the upper bound of two vector fields is sharp. In order to do this, consider the discontinuous system defined in two sectors given by

$$
\begin{aligned}
\dot{x} &= f_1(x,y) = -y - x^2,\\
\dot{y} &= g_1(x,y) = x - xy,
\end{aligned}
$$

if $y \geq 0$ and

$$
\begin{aligned}
\dot{x} &= f_2(x,y) = -y - x\left(-\sqrt{x^2+y^2} - \frac{2}{\sqrt{x^2+y^2}} + 3\right),\\
\dot{y} &= g_2(x,y) = x - y\left(-\sqrt{x^2+y^2} - \frac{2}{\sqrt{x^2+y^2}} + 3\right),
\end{aligned}
$$

if $y \leq 0$. To see this system is of Riccati type, notice that

$$\begin{aligned}
xg_1(x,y) - yf_1(x,y) &= \varphi_1(x,y) = x^2 + y^2, \\
xf_1(x,y) + yg_1(x,y) &= \psi_1(x,y) = -x\left(x^2+y^2\right), \\
xg_2(x,y) - yf_2(x,y) &= \varphi_2(x,y) = x^2 + y^2, \\
xf_2(x,y) + yg_2(x,y) &= \psi_2(x,y) = \sqrt{x^2+y^2}\left(-3\sqrt{x^2+y^2} + x^2 + y^2 + 2\right).
\end{aligned}$$

and therefore

$$\begin{aligned}
\varphi_1(\rho x, \rho y) &= \rho^2 \varphi_1(x,y), \quad \psi_1(\rho x, \rho y) = \rho^3 x\left(x^2+y^2\right), \\
\varphi_2(\rho x, \rho y) &= \rho^2 \varphi_2(x,y), \quad \psi_2(\rho x, \rho y) = \rho^3\left(x^2+y^2\right)^{3/2} - 3\rho^2\left(x^2+y^2\right) + 2\rho\sqrt{x^2+y^2},
\end{aligned}$$

so φ_i and ψ_i clearly satisfy **H.1** and **H.2** for $i = 1, 2$. Now, changing to polar coordinates and taking θ as the new independent variable, the provided system becomes

$$r' = \begin{cases} -r^2\cos\theta & \text{if} \quad 0 \leq \theta \leq \pi, \\ 2 - 3r + r^2 & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

Solving each of the previous differential equations and composing the respective transition functions, we can obtain the Poincaré map which writes

$$\Pi(r_0) = \frac{(e^\pi - 2)\,r_0 - 2e^\pi + 2}{(e^\pi - 1)\,r_0 - 2e^\pi + 1}.$$

As the Poincaré map has two fixed points, namely $r_0 = 1$ and $r_0 = 2$ and they are isolated, we obtain two limit cycles for the considered Riccati systems, so Theorem 3 is proved. □

In figure 3 we can see the two limit cycles that exist for the system presented in the last proof. They correspond to the solutions

$$\gamma(\theta) = \begin{cases} \dfrac{1}{\sin(\theta)+1} & \text{if} \quad 0 \leq \theta \leq \pi, \\ 1 & \text{if} \quad \pi \leq \theta \leq 2\pi, \end{cases}$$

$$\alpha(\theta) = \begin{cases} \dfrac{1}{\sin(\theta)+\frac{1}{2}} & \text{if} \quad 0 \leq \theta \leq \pi, \\ 2 & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

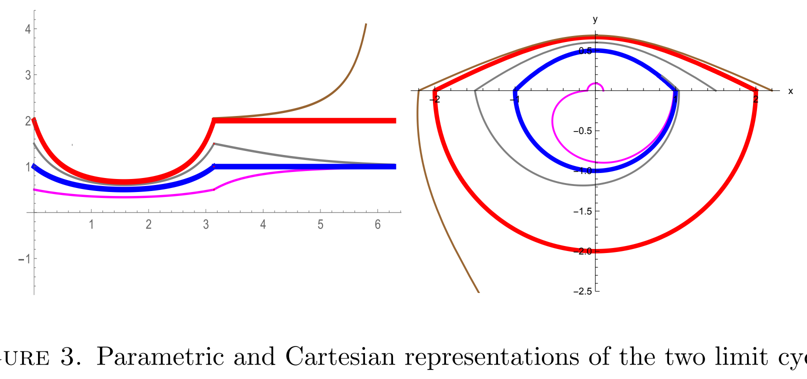

FIGURE 3. Parametric and Cartesian representations of the two limit cycles.

In the proof of the upper bound in Theorem 3 we do not provide explicitly the expression of $\Pi(r)$ in terms of the functions f_i and g_i of system (2). The main reason is because it involves $4k$ integrals en terms of arbitrary angles and complicated expressions of the coefficients representing system (2) in its polar differential equation's version, see the proof of Theorem 2. It escapes from the main goals of the paper.

We provide, instead, a *posteriori* analysis about the hyperbolicity of the eventual obtained limit cycles. Actually, it is not difficult to obtain from the expression of $\Pi(r)$ in (5) that system (2) has no limit cycles if

$$\frac{d-a+\sqrt{(a-d)^2+4bc}}{2c}<0,$$

one limit cycle if

$$\frac{d-a+\sqrt{(a-d)^2+4bc}}{2c}>0 \text{ and} \frac{d-a-\sqrt{(a-d)^2+4bc}}{2c}<0$$

and two limit cycles provided that

$$\frac{d-a-\sqrt{(a-d)^2+4bc}}{2c}>0.$$

Moreover, if a limit cycle does exist, then it is hyperbolic if neither $(a-d)^2+4bc=0$ nor $b=d-a=0$.

Next we prove Theorems 4 and 5. They address polynomial Riccati systems so a finer analysis is achievable. Despite of that, as in Theorem 3, the hiperbolicity cannot be explicitly stated because the Poincaré has a large and not trivial expression. A approach *a posteriori* as done before can also be done for Theorems 4 and 5.

Proof of Theorem 4. Assume that system (2) is of Riccati type being f_i and g_i as stated in Theorem 4 for $j=1,\ldots,k$. We notice that, for each $j=1,\ldots,k$, the functions

$$\varphi_j(x,y)=xg_j(x,y)-yf_j(x,y)=q^j_{00}x-p^j_{00}y+q^j_{10}x^2+(q^j_{01}-p^j_{10})xy-p^j_{01}y^2,$$

consequently

$$\varphi_j(\rho x,\rho y)=\rho(q^j_{00}x-p^j_{00}y+q^j_{10}\rho x^2+(q^j_{01}-p^j_{10})\rho xy-p^j_{01}\rho y^2).$$

The functions φ_j can be homogeneous of degree one or two. So, in the first case, φ is homogeneous of degree one if, and only if, the parameters satisfy $p^j_{01}=q^j_{10}=0$ and $p^j_{10}=q^j_{01}$. In the second case, φ is homogeneous of degree two if, and only if, the parameters satisfy $p^j_{00}=q^j_{00}=0$. Consequently, in both cases hypothesis **H.1** is fulfilled. One can see that under these conditions we obtain the families $\mathcal{L}^j_1$ and $\mathcal{L}^j_2$ and moreover they fulfill the conditions of hypothesis **H.2**.

Now we prove bullets (i), (ii) and (iii) of Theorem 4. Because the origin is a monodromic point of system (2), we can apply Theorem 3 to guarantee an upper bound of at most two limit cycles for system (2). However, writing the expressions of family $\mathcal{L}^j_1$ e $\mathcal{L}^j_2$ in polar coordinates and calculating the associated transitions maps Π^j_1 and Π^j_2 we obtain, respectively,

$$\Pi^j_1(r_0)=\frac{q^j_{00}r_0}{q^j_{00}\cos(\theta_j)-p^j_{00}\sin(\theta_j)-p^j_{10}r_0\sin(\theta_j)}, \tag{11}$$

and

$$\Pi^j_2(r_0)=\sqrt{2q^j_{10}}\,e^{\beta_j}\left(-p^j_{01}+q^j_{10}+(p^j_{01}+q^j_{10})cos(2\theta_j)+(-p^j_{10}+q^j_{01})sin(2\theta_j)\right)^{-\frac{1}{2}}r_0.$$

where

$$\beta_j=-\frac{(p^j_{10}+q^j_{01})\left(arctanh\left(\frac{p^j_{10}-q^j_{01}}{\sqrt{(p^j_{10}-q^j_{01})^2+4p^j_{01}q^j_{10}}}\right)+arctanh\left(\frac{-p^j_{10}+q^j_{01}-2p^j_{01}tan(\theta_j)}{\sqrt{(p^j_{10}-q^j_{01})^2+4p^j_{01}q^j_{10}}}\right)\right)}{\sqrt{(p^j_{10}-q^j_{01})^2+4p^j_{01}q^j_{10}}}.$$

As the Poincaré map is the composition of transition maps of the form Π_1^j and Π_2^j, being at least one of them of the form Π_1^j by hypothesis, we obtain that the Poincaré map writes

$$\Pi(r_0) = \frac{\gamma_1 r_0}{\gamma_2 + \gamma_3 r_0},$$

where the terms γ_1, γ_2 and γ_3 depend on the coefficients of vector field and of angles θ_j. Therefore, there exist at most one nonzero fixed point and then bullet (i) is proved.

To prove bullet (ii), assume that system (2) has at least a subsystem of the family $\mathcal{L}_1^j$ for some j and $k = 1$ or $k = 2$. If $p_{10}^j = 0$ the trajectories of this family are parallel straight lines, otherwise it presents a unique equilibrium point which is a star node. In both case it is easy to see that no return map can be defined because either there is no switching manifold or it is composed by a single straight line. But that is a contradiction to the fact that the origin is a monodromic point, and therefore we get $k \geq 3$.

In order to prove bullet (iii), take $\widetilde{k} = 4$, $\Sigma = \cup_{i=1}^4 R_i$ where R_i are the half axes of the Cartesian plane in such way that the sectors S_i is the $i-$th quadrant, $i = 1, \ldots, 4$. In this case, the angles θ_i for $i = 1, 2, 3, 4$, are, respectively, $\pi/2$, π, $3\pi/2$ and 2π. Now set in S_i the family $\widetilde{\mathcal{L}}_2^i$ satisfying that $p_{00}^1, p_{00}^2, q_{00}^2, q_{00}^3 < 0$ and all other parameters are positive. Under this condition the origin is a monodromic regular point.

Following the proof of bullet (i), there exist four transition maps $\Pi_1^j(r_0)$, $i = 1, \ldots, 4$ which writes

$$\Pi_1^2(r_0) = -\Pi_1^4(r_0) = r, \quad \Pi_1^1(r_0) = \frac{q_{00}^1 r}{-p_{00}^1 - p_{10}^1 r} \quad \text{and} \quad \Pi_1^3(r_0) = \frac{q_{00}^3 r}{-p_{00}^3 - p_{10}^3 r},$$

which are of Möbius transformations. The Poincaré map is then obtained by the composition $\Pi = \Pi_1^4 \circ \Pi_1^3 \circ \Pi_1^2 \circ \Pi_1^1$ and it is also a Möbius transformations which writes

$$\Pi(r_0) = \frac{q_{00}^1 q_{00}^3 r}{p_{00}^1 p_{00}^3 + (q_{00}^1 p_{10}^3 + p_{00}^3 p_{10}^1) r}.$$

The nonzero fixed point in this case is

$$\widetilde{r}_0 = \frac{-p_{00}^1 p_{00}^3 + q_{00}^1 q_{00}^3}{q_{00}^1 p_{10}^3 + p_{00}^3 p_{10^1}}$$

which is positive by hypothesis. That ends the proof of bullet (iii) and Theorem 4. □

We remark that in the proof of bullet (i) in Theorem 4, if every system forming system (2) belongs to the family $\mathcal{L}_2$, then the composition of the $k-$transition functions Π_2 is of the form kr_0 so the unique fixed point is the origin. Clearly, in such case no periodic orbit can occur.

Proof of Theorem 5. The proof of the first part of the theorem which provides the two families $\mathcal{Q}_1$ and $\mathcal{Q}_2$ involves some tedious calculations but it is essentially similar to the proof of Theorem 4, so we will omit it here.

The proof of bullet (i) follows directly from Theorem 3. Finally, the proof of bullet (ii) follows from the application provided in Subsection 3.3. □

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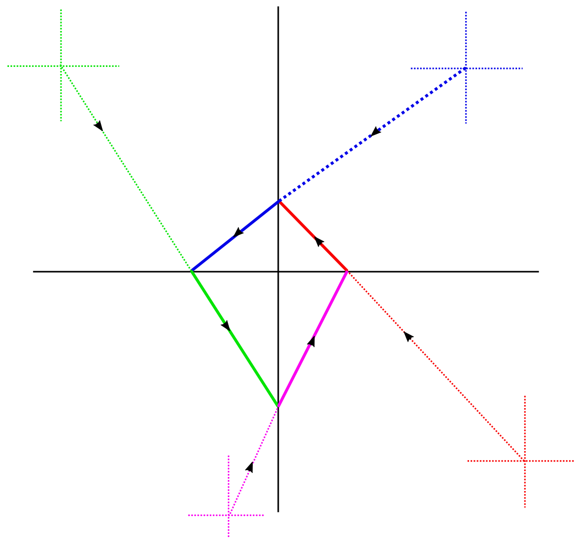

FIGURE 4. A limit cycle formed by straight lines solutions of families from $\widetilde{\mathcal{L}_2^1}$ to $\widetilde{\mathcal{L}_2^4}$.

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