ZERO-HOPF BIFURCATION IN A CHUA SYSTEM

RODRIGO D. EUZÉBIO[1,2] AND JAUME LLIBRE[2]

Abstract. A zero-Hopf equilibrium is an isolated equilibrium point whose eigenvalues are $\pm\omega i \neq 0$ and 0. In general for a such equilibrium there is no theory for knowing when from it bifurcates some small–amplitude limit cycles moving the parameters of the system. Here we study the zero-Hopf bifurcation using the averaging theory. We apply this theory to a Chua system depending on 6 parameters, but the way followed for studying the zero-Hopf bifurcation can be applied to any other differential system in dimension 3 or higher.

In this paper first we show that there are three 4-parameter families of Chua systems exhibiting a zero-Hopf equilibrium. After, by using the averaging theory, we provide sufficient conditions for the bifurcation of limit cycles from these families of zero-Hopf equilibria. From one family we can prove that 1 limit cycle bifurcates, and from the other two families we can prove that 1, 2 or 3 limit cycles bifurcate simultaneously.

1. Introduction and statement of the main results

The Chua system is a classical model for electronic circuit and one of the most simplest models presenting chaos. It was presented by Chua, Komuro and Matsumoto [13] in 1986 and exhibit a rich range of dynamical behaviour. There are several different models of Chua's systems see for instance [20, 19, 25, 31, 35].

The Chua circuit considered in [13] is a relaxation oscillator with a cubic nonlinear characteristic. It can be though as a circuit comprising a harmonic oscillator for which the operation is based on a field-effect transistor, coupled to a relaxation oscillator composed of a tunnel diode. The Chua system can be described by the following equations

$$
\begin{aligned}
\frac{dx}{dt} &= a(z - bx - a_2x^2 - a_1x^3),\\
\frac{dy}{dt} &= -z,\\
\frac{dz}{dt} &= -b_1x + y + b_2z.
\end{aligned}
\tag{1}
$$

Note that it depends on six parameters a, a_1, a_2, b, b_1 and b_2.

In [27] the autors analyze the existence of local and global analytic first integrals in the Chua system. In [32] the authors use techniques of Differential Geometry in order to obtain an analytical expression of the slow manifold equation of Chua system. In [28] was studied the dynamics at infinity of the Chua system for the particular case where b_1 and b_2 are both one. Besides, we can found some aspects about the Hopf bifurcation in [29] and [3]. In this paper, by using averaging theory,

2010 *Mathematics Subject Classification.* 37G15, 37G10, 34C07.

Key words and phrases. Chua system, periodic orbit, averaging theory, zero Hopf bifurcation.

we study the limit cycles that can bifurcate from zero-Hopf equilibrium points of the Chua system (1) . We note that at these points for our system (1) we can apply neither the classical Hopf bifurcation theory which needs that the real eigenvalue be non-zero, nor the standard theory developed up to now for some special cases of zero-Hopf equilibrium points as the ones analyzed in the papers [2, 14, 17, 21]. A possible approach for studying the zero-Hopf equilibrium is to pass to normal form, but this needs some work. Here we shall show how to study zero-Hopf bifurcations directly without needing to pass to normal form. Other authors also have studied the zero-Hopf bifurcation in other Chua systems different from the Chua system here analyzed, see for instance the articles [3, 29, 30] where $\dot{y} = x - y - z$, among other differences with the system (1).

The Chua system (1) can have at most three equilibria, namely: the origin and the two equilibria

$$p_{\pm} = \left(\frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 b}}{2a_1}, -\frac{a_2 b_1}{2a_1} \pm \frac{b_1\sqrt{a_2^2 - 4a_1 b}}{2a_1}, 0 \right),$$

if $a_2^2 - 4a_1 b > 0$ and $a_1 \neq 0$. When $a_2^2 - 4a_1 b = 0$ and $a_1 a_2 \neq 0$ the system has only two equilibra, the origin and the equilibrium

$$p = \left(\frac{-a_2}{2a_1}, -\frac{a_2 b_1}{2a_1}, 0 \right).$$

Otherwise the origin is the unique equilibrium of the system.

As far as we know, the study of existence or non-existence of zero-Hopf equilibria and zero-Hopf bifurcation in the Chua system have not been considered in the literature. In this paper we have this objective. The method used here for studying the zero-Hopf bifurcation can be applied to any differential system in $\mathbb{R}^3$.

A *zero-Hopf equilibrium* is an equilibrium point of a $3-$dimensional autonomous differential system which has a zero eigenvalue and a pair of purely imaginary eigenvalues. In general the zero-Hopf bifurcation is a $2-$parameter unfolding of a $3-$dimensional autonomous differential system with a zero-Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalue if the two parameters take zero values and the unfolding has different dynamics in a small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. To read more about zero-Hopf bifurcation, see Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in [16], [17], [18], [22] and [34]. Moreover, complex phenomena can occur at an isolated zero-Hopf equilibrium, as bifurcation of complicated invariant sets of the unfolding and a local birth of "chaos", as can be seen in the work of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, Scheurle and Marsden in [6], [7], [10], [12] and [34].

In the next proposition we characterize the Hopf equilibria of the Chua system.

Proposition 1. *There are three* 4*-parameter families of Chua systems having a zero-Hopf equilibrium point, one for the equilibrium point located at the origin and the other two for each one of the equilibria* $p_{\pm}$ *when they exist. Namely,*

(a) $b = b_2 = 0$ *and* $ab_1 + 1 > 0$ *for the origin; and*
(b) $b = a_2^2/(4a_1)$*,* $b_2 = 0$*,* $ab_1 + 1 > 0$*,* $a_2^2 - 4a_1 b > 0$ *and* $a_1 \neq 0$ *for* $p_{\pm}$*.*

The next result gives sufficient conditions for the bifurcation of a limit cycle from the origin when it is a zero-Hopf equilibrium.

Theorem 2. *Let*

$$(a, a_1, a_2, b, b_1, b_2) = (\overline{a}_0 + \varepsilon\alpha_0, \overline{a}_1 + \varepsilon\alpha_1, \overline{a}_2 + \varepsilon\alpha_2, \varepsilon\beta_0, \frac{\omega^2 - 1}{a} + \varepsilon\beta_1, \varepsilon\beta_2).$$

If $\overline{a}_0\overline{a}_2 \neq 0$, $|\omega| \neq 0, 1$ *and*

$$\Gamma = (\overline{a}_0\beta_0(1-\omega^2) + \beta_2\omega^2)(\overline{a}_0\beta_0\omega^2(1-\omega^2) + \beta_2\omega^4) > 0,$$

then for $\varepsilon > 0$ *sufficiently small the Chua system has a zero-Hopf bifurcation at the equilibrium point located at the origin of coordinates, and a limit cycle appears at this equilibrium when* $\varepsilon = 0$. *Moreover, this limit cycle has the same kind of stability or instability than an equilibrium point of a planar differential system with eigenvalues*

$$\frac{-\beta_2\omega^5 \pm \sqrt{\omega^6(\beta_2^2\omega^4(3-2\omega^2) + 2\overline{a}_0^2\beta_0^2(\omega^2-1)^3)}}{2\omega^6(\omega^2-1)}. \tag{2}$$

The following result provides sufficient conditions for the bifurcation of a limit cycle from the equilibrium p_- when it is zero-Hopf equilibrium.

Theorem 3. *Consider the vector* $(a, a_1, a_2, b, b_1, b_2)$ *given by*

$$\begin{aligned} a &= \overline{a}_0 + \varepsilon\alpha_0 + \varepsilon^2\xi_0, \\ a_1 &= \overline{a}_1 + \varepsilon\alpha_1 + \varepsilon^2\xi_1, \\ a_2 &= \varepsilon\alpha_2 + \varepsilon^2\xi_2, \\ b &= \frac{a_2^2}{4a_1} + \varepsilon^2\zeta_0, \\ b_1 &= \frac{\omega^2-1}{a} + \varepsilon\beta_1 + \varepsilon^2\zeta_1, \\ b_2 &= \varepsilon^2\zeta_2. \end{aligned} \tag{3}$$

If $\overline{a}_1\omega \neq 0$ *and* $\overline{a}_1\zeta_0 < 0$ *then, for* $\varepsilon > 0$ *sufficiently small the Chua system has a zero-Hopf bifurcation at the equilibrium point located at* p_- *and three limit cycles can bifurcate from this equilibrium when* $\varepsilon = 0$. *Moreover, examples of systems where* 1, 2 *or* 3 *limit cycles bifurcate simultaneously are given.*

Proposition 1 and Theorems 2 and 3 are proved in section 3. In particular, booth theorems are proved using the averaging method. This method will be briefly summarized in the next section. We note that Theorem 2 is proved using averaging theory of first order, but the proof of Theorem 3 needs averaging of second order.

Also the stability or instability of the bifurcated limit cycles in Theorem 3 can be studied, but the expressions of the eigenvalues which provide such stability or instability are huge and we do not give them here.

Remark 1. *For the equilibrium point* p_+ *we have analogous results to the ones of Theorem 3 for* p_-. *For this reason, we omit the statement of the result for the equilibrium* p_+ *and its proof. This fact is not due to any symmetry of the Chua system, it is only due to the fact that doing the corresponding computations for the equilibrium point* p_+ *we obtain the same results than in the equilibrium* p_-.

2. Limit cycles via averaging theory

The averaging method is a classical tool in nonlinear analysis and dynamical systems. The procedure of averaging can be found already in the work of Lagrange [23] and Laplace [24] who provided an intuitive justification of the method. After them, Poincaré considered the determination of periodic solutions by series expansion with respect to a small parameter, but until around 1930 we see the start of precise statements and proofs in averaging theory. After this time many new results in the theory of averaging have been obtained. The main contribution in direction to the formalization of the method started with Appleton and van der Pol [4] and Fatou [15], and later with the work of Bogoliubov and Krylov [9] and Bogoliubov [8].

Now we present the basic results on the averaging theory of first and second order. The averaging of first order for studying periodic orbits can be found in [33], see Theorems 11.5 and 11.6. It can be summarized as follows.

Theorem 4. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{4}$$

and

$$\dot{y} = \varepsilon f^0(y), \qquad y(0) = x_0. \tag{5}$$

where $x, y, x_0 \in \Omega$ *an open subset of* $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f *and* g *are periodic of period* T *in the variable* t, *and* $f^0(y)$ *is the averaged function of* $f(t,x)$ *with respect to* t, *i.e.,*

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{6}$$

Suppose:

(i) f, *its Jacobian* $\dfrac{\partial f}{\partial x}$, *its Hessian* $\dfrac{\partial^2 f}{\partial x^2}$, g *and its Jacobian* $\dfrac{\partial g}{\partial x}$ *are defined, continuous and bounded by a constant independent on* ε *in* $[0,\infty) \times \Omega$ *and* $\varepsilon \in (0,\varepsilon_0]$;

(ii) T *is a constant independent of* ε; *and*

(iii) $y(t)$ *belongs to* Ω *on the interval of time* $[0, 1/\varepsilon]$. *Then the following statements hold.*

  (a) *On the time scale* $1/\varepsilon$ *we have that* $x(t) - y(t) = O(\varepsilon)$, *as* $\varepsilon \to 0$.

  (b) *If* p *is a singular point of the averaged system* (5) *such that the determinant of the Jacobian matrix*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \tag{7}$$

  is not zero, then there exists a limit cycle $\phi(t,\varepsilon)$ *of period* T *for system* (4) *which is close to* p *and such that* $\phi(0,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

  (c) *The stability or instability of the limit cycle* $\phi(t,\varepsilon)$ *is given by the stability or instability of the singular point* p *of the averaged system* (5). *In fact, the singular point* p *has the stability behaviour of the Poincaré map associated to the limit cycle* $\phi(t,\varepsilon)$.

The next result present the second order averaging method of a periodic differential system. For a proof see Theorem 3.5.1 of Sanders and Verhulst in [33], see also [11].

Theorem 5. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad x(0) = x_0 \tag{8}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2(f^{10}(y) + g^0(y)), \quad y(0) = x_0, \tag{9}$$

with $f, g : [0, \infty) \times \Omega \to \mathbb{R}^n$, $R : [0, \infty) \times \Omega \times (0, \varepsilon_0] \to \mathbb{R}^n$, Ω *an open subset of* $\mathbb{R}^n$, f, g *and* R *periodic of period* T *in the variable* t,

$$f^1(t, x) = \frac{\partial f}{\partial x} y^1(t, x), \quad \text{where} \quad y^1(t, x) = \int_0^t f(s, x) ds.$$

Of course, f^0, f^{10} *and* g^0 *denote the averaged functions of* f, f^1 *and* g, *respectively, defined as in* (6). *Suppose:*

- (i) $\partial f/\partial x$ *is Lipschitz in* x, g *and* R *are Lipschitz in* x *and all these functions are continuous on their domain of definition;*
- (ii) $|R(t, x, \varepsilon)|$ *is bounded by a constant uniformly in* $[0, L/\varepsilon) \times \Omega \times (0, \varepsilon_0]$*;*
- (iii) T *is a constant independent of* ε*; and*
- (iv) $y(t)$ *belongs to* Ω *on the interval of time* $[0, 1/\varepsilon]$. *Then the following statements hold.*
  - (a) *In the time scale* $1/\varepsilon$ *we have that* $x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2)$.
  - (b) *If* $f^0(y) \equiv 0$ *and* p *is a singular point of averaged system* (9) *such that*

    $$\left.\frac{\partial(f^{10} + g^0)(y)}{\partial y}\right|_{y=p}$$

    is not zero, then there exist a limit cycle $\phi(t, \varepsilon)$ *of period* T *for system* (8) *which is close to* p *and such that* $\phi(0, \varepsilon) \to p$ *as* $\varepsilon \to 0$.
  - (c) *The stability or instability of the limit cycle* $\phi(t, \varepsilon)$ *is given by the stability os instability of the singular point* p *of the averaged system* (9). *In fact, the singular point* p *has the stability behaviour of the Poincaré map associated to the limit cycle* $\phi(t, \varepsilon)$.

3. Proofs

In this section we give the proofs of the results presented in section 1.

Proof of Proposition 1. The characteristic polynomial of the linear part of the Chua system at the origin is

$$p(\lambda) = -\lambda^3 + (b_2 - ab)\lambda^2 + (b_2 ab - ab_1 - 1)\lambda - ab.$$

Imposing that $p(\lambda) = -\lambda(\lambda^2 + \omega^2)$, we obtain $b = b_2 = 0$ and $b_1 = (\omega^2 - 1)/a$. So statement (a) follows.

The characteristic polynomial of the linear part of the Chua system at p_- is given by

$$p(\lambda) = -\frac{(1 - b_2\lambda + \lambda^2)[2a_1\lambda + a(a_2^2 + a_2\sqrt{a_2^2 - 4a_1 b} - 4a_1 b)] + 2a_1 b_1 a\lambda}{2a_1}$$

The proposition follows imposing that $p(\lambda) = -\lambda(\lambda^2+\omega^2)$, and that the equilibrium point p_- exists. □

Proof of Theorem 2. If we consider

$$(a, a_1, a_2, b, b_1, b_2) = (\overline{a}_0 + \varepsilon\alpha_0, \overline{a}_1 + \varepsilon\alpha_1, \overline{a}_2 + \varepsilon\alpha_2, \varepsilon\beta_0, \frac{\omega^2-1}{a} + \varepsilon\beta_1, \varepsilon\beta_2).$$

with $\varepsilon > 0$ a sufficiently small parameter, then the Chua system becomes

$$\begin{aligned} \dot{x} &= (\overline{a}_0 + \varepsilon\alpha_0)(\varepsilon\beta_0 x + z - (\overline{a}_2 + \varepsilon\alpha_2)x^2 - (\overline{a}_1 + \varepsilon\alpha_1)x^3), \\ \dot{y} &= -z, \\ \dot{z} &= -\left(\varepsilon\beta_1 + \frac{\omega^2-1}{\overline{a}_0 + \varepsilon\alpha_0}\right)x + y + \varepsilon\beta_2 z. \end{aligned} \tag{10}$$

By the rescaling of variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, system (10) becomes

$$\begin{aligned} \dot{X} &= (\overline{a}_0 + \varepsilon\alpha_0)(\varepsilon\beta_0 X + Z - \varepsilon(\overline{a}_2 + \varepsilon\alpha_2)X^2 - \varepsilon^2(\overline{a}_1 + \varepsilon\alpha_1)X^3), \\ \dot{Y} &= -Z, \\ \dot{Z} &= -\left(\varepsilon\beta_1 + \frac{\omega^2-1}{\overline{a}_0 + \varepsilon\alpha_0}\right)X + Y + \varepsilon\beta_2 Z. \end{aligned} \tag{11}$$

Now we shall write the linear part at the origin of (11) into its real Jordan normal form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

when $\varepsilon = 0$. For doing that we do the linear change of variables $(X, Y, Z) \to (u, v, w)$ given by

$$\begin{aligned} X &= \frac{\overline{a}_0(w + \omega v)}{\omega^2}, \\ Y &= w - \frac{w}{\omega^2} - \frac{v}{\omega}, \\ Z &= u. \end{aligned} \tag{13}$$

In these new variables, system (11) is written as follow

$$
\begin{aligned}
\dot{u} = &\quad -v\omega + \varepsilon\frac{-(\alpha_0(1-\omega^2)+\overline{a}_0^2\beta_1)(w+\omega v)+\overline{a}_0 u\beta_2\omega^2}{\overline{a}_0\omega^2} \\
&\quad -\varepsilon^2\frac{\alpha_0^2(\omega^2-1)(w+\omega v)}{\overline{a}_0^2\omega^2}, \\
\dot{v} = &\quad u\omega - \varepsilon\frac{(\omega^2-1)(-u\alpha_0\omega^4+\overline{a}_0^2(w+\omega v)(\beta_0\omega^2+\overline{a}_0\overline{a}_2(w+\omega v)))}{\overline{a}_0\omega^5} \\
&\quad -\varepsilon^2\frac{1}{\omega^7}(\omega^2-1)(w+\omega v)(\alpha_0\beta_0\omega^4+\overline{a}_0(w+\omega v)(\overline{a}_2\alpha_0\omega^2 \\
&\quad +\overline{a}_0\alpha_2\omega^2+\overline{a}_0^2\alpha_1(w+\omega v))), \\
\dot{w} = &\quad -\varepsilon\frac{-u\alpha_0\omega^4+\overline{a}_0^2(w+\omega v)(\beta_0\omega^2+\overline{a}_0\overline{a}_2(w+\omega v))}{\overline{a}_0\omega^4} \\
&\quad -\varepsilon^2\frac{(w+\omega v)(\alpha_0\beta_0\omega^4+\overline{a}_0(w+\omega v)(\overline{a}_2\alpha_0\omega^2+\overline{a}_0\alpha_2\omega^2))}{\omega^6}.
\end{aligned} \tag{14}
$$

Writing the differential system (14) in cylindrical coordinates (r,θ,w) by $u = r\cos\theta$, $v = r\sin\theta$ and $w = w$ we have

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon\Bigg(\frac{r\beta_2\cos^2\theta}{\omega} - \frac{\overline{a}_0(\omega^2-1)\sin\theta(w+r\omega\sin\theta)(\overline{a}_0\overline{a}_2 w+\beta_0\omega^2}{\omega^6} \\
&\quad \frac{+\overline{a}_0\overline{a}_2 rw\sin\theta)}{\omega^6} - \frac{\cos\theta(w(\alpha_0+\overline{a}_0^2\beta_1-\alpha_0\omega^2)+rw(\overline{a}_0\beta_1}{\overline{a}_0\omega^3} \\
&\quad \frac{-2\alpha_0(\omega^2-1))\sin\theta)}{\overline{a}_0\omega^3}\Bigg) + O(\varepsilon^2), \\
\frac{dw}{d\theta} &= \varepsilon\frac{r\alpha_0\omega^4\cos\theta - \overline{a}_0^2(w+r\omega\sin\theta)(\overline{a}_0\overline{a}_2 w+\beta_0\omega^2+\overline{a}_0\overline{a}_2 r\omega\sin\theta)}{\overline{a}_0\omega^5} \\
&\quad +O(\varepsilon^2).
\end{aligned} \tag{15}
$$

Now we apply the first order averaging theory as described in Theorem 4 of section 2. In order to do this, we note that (15) satisfies all the assumptions of Theorem 4, where we identify $t = \theta$, $T = 2\pi$, $x = (r,w)^T$, $F(\theta,r,w) = (F_1(\theta,r,w), F_2(\theta,r,w))$ and $f(r,w) = (f_1(r,w), f_2(r,w))$. In short system (15) is the normal form of system (1) in order to apply the averaging theory.

By calculating f_1 and f_2, we get

$$
\begin{aligned}
f_1(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,w)d\theta \\
&= \frac{r(\beta_2\omega^4 - 2\overline{a}_0^2\overline{a}_2 w(\omega^2-1) - \overline{a}_0\beta_0\omega^2(\omega^2-1))}{2\omega^5}, \\
f_2(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,w)d\theta \\
&= -\frac{\overline{a}_0(2w\beta_0\omega^2+\overline{a}_0\overline{a}_2(2w^2+r^2\omega^2))}{2\omega^5}.
\end{aligned}
$$

There is only one solution (r^*, w^*) for $f_1(r,w) = f_2(r,w) = 0$ satisfying $r^* > 0$ and this solution is

$$r^* = \sqrt{\frac{\Gamma}{2\overline{a}_0^4\overline{a}_2^2(\omega^2-1)^2}},$$
$$w^* = \frac{\overline{a}_0\beta_0\omega^2(1-\omega^2)+\beta_2\omega^4}{2\overline{a}_0^2\overline{a}_2(\omega^2-1)},$$

since $\overline{a}_0\overline{a}_2 \neq 0$, $|\omega| \neq 1$ and $\Gamma > 0$. We recall that Γ is defined in the statement of Theorem 2.

We note that the Jacobian (7) at (r^*, w^*) takes the value

$$\frac{\beta_2^2\omega^4 - \overline{a}_0^2\beta_0^2(\omega^2-1)^2}{2\omega^6(\omega^2-1)}$$

and the eigenvalues of the Jacobian matrix

$$\left.\frac{\partial(f_1,f_2)}{\partial(r,w)}\right|_{(r,w)=(r^*,w^*)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}\omega^5}\sqrt{\overline{a}_0(\omega^2-1)\Gamma} \\ -\frac{1}{\omega^3}\sqrt{\frac{\overline{a}_0\Gamma}{2(\omega^2-1)^3}} & \frac{\beta_2}{\omega(1-\omega^2)} \end{pmatrix}$$

are the ones given in (2).

In short, from Theorem 4 we conclude the proof once we show that periodic solutions corresponding to (r^*, w^*) provides a periodic solution bifurcating from the origin of coordinates of the differential system (10) when $\varepsilon = 0$. Theorem 4 guarantees for $\varepsilon > 0$ sufficiently small the existence of a periodic solution corresponding to the point (r^*, w^*) of the form $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ such that $(r(0,\varepsilon), w(0,\varepsilon)) \to (r^*, w^*)$ when $\varepsilon \to 0$. So system (14) has a periodic solution

$$(u(\theta,\varepsilon) = r(\theta,\varepsilon)\cos\theta, v(\theta,\varepsilon) = r(\theta,\varepsilon)\sin\theta, w(\theta,\varepsilon)) \tag{16}$$

for $\varepsilon > 0$ sufficiently small. Consequently, from relation (16) through the linear change of variables (13) system (11) has a periodic solution $(X(\theta), Y(\theta), Z(\theta))$. Finally, for $\varepsilon > 0$ sufficiently small system (10) has a periodic solution $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon \to 0$. Thus, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of theorem. □

Since the proof of Theorem 3 is very similar to the of Theorem 2, then we will omit some steps in order to avoid some long expressions.

Proof of Theorem 3. Suppose that we have the conditions given in (3) on the parameters of Chua system (1). Then, by a translation of the equilibrium point p_- at the origin of coordinates, and a rescaling of variables given by $(x,y,z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$ the Chua system becomes

$$\begin{aligned} \dot{X} &= A_1X + (\overline{a}_0 + \alpha_0\varepsilon + \varepsilon^2\xi)Z + A_2X^2 + A_3X^3, \\ \dot{Y} &= -Z, \\ \dot{Z} &= A_4X + Y + \varepsilon^2\zeta_2 Z, \end{aligned} \tag{17}$$

where

$$\begin{aligned}
A_1 &= \varepsilon^2(1/2\overline{a}_1^2)(\overline{a}_0(-4\overline{a}_1^2\zeta_0 - \alpha_1\alpha_2\varepsilon\sqrt{-\overline{a}_1\zeta_0} + 2\overline{a}_1\sqrt{-\overline{a}_1\zeta_0}(\alpha_2+\varepsilon\xi_2)\\
&\quad +2\overline{a}_1\alpha_0\varepsilon(-2\overline{a}_1\zeta_0+\alpha_2\sqrt{-\overline{a}_1\zeta_0})),\\
A_2 &= \varepsilon^2(1/2\overline{a}_1)(\overline{a}_0(3\alpha_1\varepsilon\sqrt{-\overline{a}_1\zeta_0}+\overline{a}_1(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0}+\varepsilon\xi_2))\\
&\quad +\overline{a}_1\alpha_0\varepsilon(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})),\\
A_3 &= \varepsilon^2(\overline{a}_1\alpha_0\varepsilon+\overline{a}_0(\overline{a}_1+\alpha_1\varepsilon)),\\
A_4 &= (\omega^2-1)[\overline{a}_0^3-\alpha_0^3\varepsilon^3-\overline{a}_0^2\varepsilon(\alpha_0+\varepsilon\xi_0+\overline{a}_0\alpha_0\varepsilon^2(\alpha_0+2\varepsilon\xi_0))]\\
&\quad +\overline{a}_0^4\varepsilon(\beta_1+\varepsilon\zeta_1).
\end{aligned}$$

The linear part of (17) at p_- in the real Jordan normal form when $\varepsilon = 0$ is given by (12), and doing also the linear change of variables $(X, Y, Z) \to (u, v, w)$ given by (13) we write the linear part of system (17) in its real Jordan normal form when $\varepsilon = 0$, we obtain the system

$$\begin{aligned}
\dot{u} &= -\omega v + \varepsilon(B_1 v + B_2 w) + \varepsilon^2\zeta_2 u,\\
\dot{v} &= \omega u + \frac{\varepsilon\alpha_0(\omega^2-1)u}{\overline{a}_0\omega} + \varepsilon\frac{\omega^2-1}{\overline{a}_0\omega}B_3,\\
\dot{w} &= \frac{\varepsilon\alpha_0}{\overline{a}_0}u + \frac{\varepsilon^2}{\overline{a}_0}B_3,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
B_1 &= -\frac{\overline{a}_0^3(\beta_1+\zeta_1)-\overline{a}_0(\alpha_0+\varepsilon\xi_0)(\omega^2-1)-\varepsilon\alpha_0^2(\omega^2-1)}{\overline{a}_0^2\omega},\\
B_2 &= -\frac{\overline{a}_0^3(\beta_1+\zeta_1)-\overline{a}_0(\alpha_0+\varepsilon\xi_0)(\omega^2-1)+\varepsilon\alpha_2^2(\omega^2-1)}{\overline{a}_0^2\omega^2},\\
B_3 &= \xi_0 u - \frac{\overline{a}_0^2(w+\omega v)(2(-2\overline{a}_1\zeta_0+\alpha_2\sqrt{\overline{a}_1\zeta_0})\omega^4-\overline{a}_0\overline{a}_1(\alpha_2}{2\overline{a}_1\omega^6}\\
&\quad \frac{+6\omega^2(w+\omega v)\sqrt{\overline{a}_1\zeta_0})+2\overline{a}_0^2\overline{a}_1^2(w+\omega v)^2)}{2\overline{a}_1\omega^6}.
\end{aligned}$$

If we write system (18) in cylindrical coordinates (r, θ, w) defined by $u = r\cos\theta$, $v = r\sin\theta$ and $w = w$, after we take as new independent variable the angle θ, and we apply to the system $dr/d\theta$ and $dw/d\theta$ that we obtain the second order averaging method described in Theorem 5, we get that the function $f = (f_1, f_2)$ is identically zero, and that the function $g = (g_1, g_2)$ is

$$\begin{aligned}
g_1(r,w) &= \frac{\pi r}{4\omega}\Bigg(4\zeta_2 + \frac{\overline{a}_0(\omega^2-1)(4\overline{a}_0\overline{a}_1 w(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2}{\overline{a}_1\omega^6}\\
&\quad +\frac{4(2\overline{a}_1\zeta_0-\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4-3\overline{a}_0^2\overline{a}_1^2(4w^2+3r^2\omega^2))}{\overline{a}_1\omega^6}\Bigg),\\
g_2(r,w) &= \frac{\overline{a}_0\pi}{2\overline{a}_1\omega^7}\left(4w(2\overline{a}_1\zeta_0-\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4-2\overline{a}_0^2\overline{a}_1^2 w(2w^2+3r^2\omega^2)\right.\\
&\quad \left.+\overline{a}_0\overline{a}_1(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2(2w^2+r^2\omega^2)\right).
\end{aligned}$$

In order to find solutions (r^*, w^*) of $g = 0$ we compute a Gröbner basis $\{b_k(r, w), k = 1, \dots, 20\}$ in the variables r and w for the set of polynomials $\{\overline{g}_1(r, w), \overline{g}_2(r, w)\}$

where $\overline{g}_1 = 4(\overline{a}_1\omega^7/\pi r)g_1$ and $\overline{g}_2 = (2\overline{a}_1\omega^7/\overline{a}_0\pi)g_2$ and then we will look for roots of b_1 and b_2. It is a known fact that the solutions of a Gröbner basis of $\{\overline{g}_1(r,w),\overline{g}_2(r,w)\}$ are the solutions of $\overline{g}_1 = 0$ and $\overline{g}_2 = 0$, consequently solutions of $g_1 = 0$ and $g_2 = 0$ as well. For more information about Gröbner basis see [1] and [26].

The Gröbner basis for the polynomials $\{\overline{g}_1(r,w),\overline{g}_2(r,w)\}$ in the variables r and w is formed by twenty polynomials. We only use two polynomials of this basis, namely,

$$\begin{aligned} G_1(r,w) &= 30(\omega^2-1)\overline{a}_0^4\overline{a}_1^3w^3 - 15(\omega^2-1)\omega^2\overline{a}_1^2\overline{a}_0^3(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})w^2 \\ &\quad +2\overline{a}_0\overline{a}_1\omega^4(-6\overline{a}_1\zeta_2\omega^2+\overline{a}_0(\alpha_2^2-42\overline{a}_1\zeta_0 \\ &\quad +15\alpha_2\sqrt{-\overline{a}_1\zeta_0}(\omega^2-1)))w + 2\omega^6(\overline{a}_1(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0}\zeta_2\omega^2) \\ &\quad +\overline{a}_0(8\overline{a}_1\alpha_2\zeta_0-\alpha_2^2\sqrt{-\overline{a}_1\zeta_0}-12(-\overline{a}_1\zeta_0)\sqrt{-\overline{a}_1\zeta_0}(\omega^2-1))) \end{aligned}$$

and

$$\begin{aligned} G_2(r,w) &= \overline{a}_0\overline{a}_1\omega^2(6\overline{a}_0\overline{a}_1w-\alpha_2\omega^2-6\omega^2\sqrt{-\overline{a}_1\zeta_0})r^2+2w(2\overline{a}_0^2\overline{a}_1^2w^2 \\ &\quad -\overline{a}_0\overline{a}_1w(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2+2(-2\overline{a}_1\zeta_0+\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4). \end{aligned}$$

Since $G_1(r,w) = G_1(w)$ is a polynomial of degree 3 in the variable w, it is clear that we can have at most three real solutions for w depending on the parameters of the zero-Hopf family. Replacing these three values of w in the second polynomial G_2 we have six solutions for r of the form $\pm r_i^*$ for $i = 1,2,3$, because $G_2(r,w)$ is of the form $P_1(w)r^2 + P_2(w)$. However, since r must be positive, we have at most three good solutions for $G_1 = 0$ and $G_2 = 0$. Consequently, we have at most three good solutions for $g = (g_1, g_2) = 0$ and then, by Theorem 5 and using the same arguments that in the proof of Theorem 2 when we go back through the changes of coordinates, we can have at most three limit cycles bifurcating from the equilibrium point p_-.

Moreover, if we consider $\alpha_2 = -6\sqrt{-\overline{a}_1\zeta_0}$, then the relations $(g_1(r,w), g_2(r,w)) = (0,0)$ provide three solutions given by

$$(r^*, w_\pm^*) = \left(\frac{2\omega}{\sqrt{15}}\sqrt{\frac{8\overline{a}_0\zeta_0(1-\omega^2)-\zeta_2}{\overline{a}_0^3\overline{a}_1(\omega^2-1)}}, \pm\frac{\omega^2}{\overline{a}_0\sqrt{\overline{a}_1}}\sqrt{-\frac{4\overline{a}_0\zeta_0(1-\omega^2)+2\zeta_2\omega^2}{5\overline{a}_0(\omega^2-1)}}\right)$$

and

$$(r^0, w^0) = \left(\frac{2\omega}{\sqrt{3}}\sqrt{\frac{4\overline{a}_0\zeta_0(1-\omega^2)+\zeta_2}{\overline{a}_0^3\overline{a}_1(\omega^2-1)}}, 0\right).$$

as long as the expressions in the square roots are positives. This shows that three limit cycles can bifurcate simultaneous from the equilibrium p_-. In a similar way we can produce examples with one, or two limit cycles bifurcating from p_-. This completes the proof of the theorem. □

Acknowledgments

We thanks to Pedro T. Cardin and Tiago de Carvalho their comments which help us to improve the presentation of this paper.

The first author is supported by the FAPESP-BRAZIL grants 2010/18015-6 and 2012/05635-1, and 2013/25828-1. The second author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014SGR568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, FEDER-UNAB-10-4E-378, and a CAPES grant 88881. 030454/2013-01 do Programa CSF-PVE.

References

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, American Mathematical Society, Graduate Studies in Mathematics, Vol. **3**, 1994.

[2] A. Algaba, E. Freire and E. Gamero, *Hypernormal form for the Hopf–zero bifurcation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8** (1998), 1857–1887.

[3] A. Algaba, M. Merino, F. Fernández-Sánchez and A. J. Rodríguez-Luis, *Hopf bifurcations and their degeneracies in Chua's equation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **21** (2011), 2749–2763.

[4] E.V. Appleton and B. van der Pol, *On a type of oscillation-hysteresis in a simple triode generator*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science Ser. 6, **43** (1922), 177–193.

[5] I. Baldomá, O. Castejón and T. M. Seara, *Exponentially small heteroclinic breakdown in the generic Hopf-zero singularity*, J. Dyn. Diff. Equations **25** (2013), 335–392.

[6] I. Baldomá and T. M. Seara, *Brakdown of heteroclinic orbits for some analytic unfoldings of the zero-Hopf singularity*, J. Nonlinear Sci. **16** (2006), 543–582.

[7] I. Baldomá and T. M. Seara, *The inner equation for generic analytic unfoldings of the zero-Hopf singularity*, Discrete Contin. Dyn. Syst. Ser. B **10** (2008), 323–347.

[8] N. N. Bogoliubov, *On some statistical methods in mathematical physics*, Izv. vo Akad. Nauk Ukr. SSR, Kiev, 1945.

[9] N. N. Bogoliubov and N. Krylov, *The application of methods of nonlinear mechanics in the theory of stationary oscillations*, Publ. 8 of Ukrainian Acad. Sic. Kiev, 1934.

[10] H. W. Broer and G. Vegter, *Subordinate Silnikov bifurcations near some singularities of vector fields having low codimension*, Ergodic Theory Din. Syst. **4** (1984), 509–525.

[11] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[12] A. R. Champneys and V. Kirk, *The entwined wiggling of homoclinic curves emerging from saddle-node/Hopf instabilities*, Physic D **195** (2004), 77–105.

[13] L. O. Chua, M. Komuro and T. Matsumoto, *The double scroll family*, IEEE Trans. Ciruits Syst. **33** (1986), 1072–1097.

[14] F. Dumortier, S. Ibáñez, H. Kokubu and C. Simó, *About the unfolding of a Hopf-zero singularity*, Discrete Contin. Dyn. Syst. **33** (2013), 4435–4471.

[15] P. Fatou, *Sur le mouvement d'un système soumis à des forces à courte période*, Bull. Soc. Math. France **56** (1928), 98–139.

[16] J. Guckenheimer, *On a codimension two bifurcation*, Lectures Notes in Math. **898** (1980), 99–142.

[17] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Revised and corrected reprint of the 1983 original*, Applied Mathematical Sciences **42**, Springer-Verlag, New York, 1990.

[18] M. Han, *Existence of periodic orbits and invariant tori in codimension two bifurcation of three dimensional systems*, J. Sys. Sci & Math. Scis. **18** (1998), 403–409.

[19] R. Kiliç, *Experimental study on impulsive synchronization between two modified Chua's circuits*, Nonlinear Anal. Real World Appl. **7** (2006), 1298–1303.

[20] E.S. Kuetche Mbe, H.B. Fotsin, J. Kengne and P. Woafo, *Parameters estimation based adaptive generalized projective synchronization (GPS) of chaotic Chua's circuit with application to chaos communication by parametric modulation*, Chaos Solitons Fractals **61** (2014), 27–37.

[21] B. Krauskopf and C. Rousseau, *Codimension-three unfolings of reflectionally symmetric planar vector fields*, Nonlinearity **10** (1997), 1115–1150.

[22] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Spring-Verlag, 3rd edition, 2004.

[23] J.L. Lagrange, *Mécanique Analytique* (2 vols.), edition Albert Blanchard, Paris, 1788.
[24] P.S. Laplace, *Oeuvres de Laplace*, Paris, Gauthier-Villars, 1878.
[25] K.W. Lee and S.N. Singh, *Robust control of chaos in Chua's circuit based on internal model principle*, Chaos Solitons Fractals **31** (2007), 1095–1107.
[26] H. Li, *Gröbner Bases in Ring Theory*, World Scientific Publishing, 2011.
[27] J. Llibre and C. Valls, *Analytic integrability of a Chua system*, J. Math. Phys. **49** (2008), no. 10, 102701, 9 pp.
[28] M. Messias, *Dynamics at infinity of a cubic Chua's system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **21** (2011), no. 1, 333–340.
[29] M. Messias, D. C. Braga and L. F. Mello, *Degenerate Hopf bifurcations in Chua's system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19** (2009), no. 2, 497–515.
[30] J.L. Moiola and L.C. Chua, *Hopf bifurcations and degeneracies in Chua's circuit–a persective from a frequency domain approach*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), 295–303.
[31] R. Riaza, *Dynamical properties of electrical circuits with fully nonlinear memristors*, Nonlinear Anal. Real World Appl. **12** (2011), 3674–3686.
[32] B. Rossetto and J.M. Ginoux, *Differential geometry and mechanics: applications to chaotic dynamical systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006), no. 4, 887–910.
[33] J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*, Appl. Math. Sci., vol. **59**, Springer, 1985.
[34] J. Scheurle and J. Marsden, *Bifurcation to quasi-periodic tori in the interaction of steady state and Hopf bifurcations*, SIAM. J. Math. Anal. **15** (1984), 1055–1074.
[35] J.J. Yan, J.S. Lin and T.L. Liao, *Synchronization of a modified Chua's circuit system via adaptive sliding mode control*, Chaos Solitons Fractals **36** (2008), 45–52.

[1] Departament de Matemática, IBILCE, UNESP, Rua Cristovao Colombo 2265, Jardim Nazareth, CEP 15.054-00, Sao José de Rio Preto, SP, Brazil
Email address: `rodrigo.euzebio@sjrp.unesp.br`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
Email address: `jllibre@mat.uab.cat`