PERIODIC SOLUTIONS OF *EL NIÑO* MODEL THROUGH THE VALLIS DIFFERENTIAL SYSTEM

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ABSTRACT. By rescaling the variables, the parameters and the periodic function of the Vallis differential system we provide sufficient conditions for the existence of periodic solutions and we also characterize their kind of stability. The results are obtained using averaging theory.

1. INTRODUCTION

The Vallis system, introduced by Vallis [10] in 1988, is a periodic non–autonomous 3–dimensional system that models the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force. Denoting by x the wind force, by y the difference of near–surface water temperatures of the east and west parts of the Pacific Ocean, and by z the average near–surface water temperature, the Vallis system is

$$
\begin{aligned}
\frac{dx}{dt} &= -ax + by + au(t), \\
\frac{dy}{dt} &= -y + xz, \\
\frac{dz}{dt} &= -z - xy + 1,
\end{aligned}
\tag{1}
$$

where $u(t)$ is some C^1 T−periodic function that describes the wind force under seasonal motions of air masses, and the parameters a and b are positive.

Although this model neglects some effects like Earth's rotation, pressure field and wave phenomena, it provides a correct description of the observed processes and recovers many of the observed properties of El Niño. The properties of El Niño phenomena are studied analytically in [9] and [10]. More precisely, in [10] it is shown that taking $u \equiv 0$, it

is possible to observe the presence of chaos by considering $a = 3$ and $b = 102$. Later on, in [9] it is proved that exists a chaotic attractor for system (1) after a Hopf bifurcation. This chaotic motion can be easily understanding if we observe that there exist a strong similarity between system (1) and Lorenz system, which becomes more clear under the replacement of z by $z + 1$ in system (1).

In [4] the authors examine the localization problem of compact invariant sets of nonlinear non–autonomous systems and apply the results to the Vallis system (1). In [3] the localization method for invariant compact sets of the autonomous dynamical system studied in [4] is generalized to the case of a nonautonomous system, and the localization problem for system (1) is solved.

In this paper we provide sufficient conditions in order that system (1) has periodic orbits, and additionally we characterize the stability of these periodic orbits. As far as we know, the study of existence of periodic orbits in the non–autonomous Vallis system has not been considered in the literature, with the exception of the Hopf bifurcation studied in [9].

We observe that the method used here for studying the periodic orbits can be applied to any periodic non–autonomous differential system. Indeed, in [5] the authors applied this method in order to prove the existence of periodic solutions in a periodic FitzHugh–Nagumo system.

This paper is organized as follows. In this section we state the main results. Next in section 2 we prove the results on the periodic solutions of the Vallis system using averaging theory. Finally, in section 3 we give a brief summary of the results that we need from averaging theory for proving our theorems.

From now on unless we say the contrary we will call

$$I = \int_0^T u(s)ds.$$

Now we state our main results.

Theorem 1. *For $I \neq 0$ and $a \neq b$ the Vallis system* (1) *has a T–periodic solution* $(x(t), y(t), z(t))$ *such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1 \right),$$

Moreover this periodic orbit is stable if $a > b$ and unstable if $a < b$.

We do not know the reliability of the Vallis model approximating the Niño phenomenon, but it seems that for the moment this is one of the best models for studying the Niño phenomenon. Accepting this reliability we can said the following.

The stable periodic solution provided by Theorem 1 says that the Niño phenomenon exhibits a periodic behavior if the T-periodic function $u(t)$ and the parameters a and b of the system satisfy that $I \neq 0$ and $a > b$. Moreover Theorem 1 states that this periodic solution lives near the point

$$(x, y, z) = \left(\frac{aI}{T(b-a)}, \frac{aI}{T(b-a)}, 1 \right).$$

Since the periodic solutions found in Theorems 3, 4 and 5 are also stable, we can provide a similar physical interpretation for them as we have done for the periodic solution of Theorem 1.

Theorem 2. *For $I \neq 0$ the Vallis system (1) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{aI}{Tb}, -\frac{aI}{Tb}, 1 \right),$$

Moreover this periodic orbit is always unstable.

Theorem 3. *For $I \neq 0$ the Vallis system (1) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, \frac{I}{T}, 1 \right),$$

Moreover this periodic orbit is always stable.

Theorem 4. *For $I \neq 0$ the Vallis system (1) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, 0, 1 \right),$$

Moreover this periodic orbit is always stable.

In what follows we consider the function

$$J(t) = \int_0^t u(s)ds.$$

and note that $J(T) = I$. So we have the following result.

Theorem 5. *Consider $I = 0$ and $J(t) \neq 0$ if $0 < t < T$. Then the Vallis system (1) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{a}{T} \int_0^T J(s)ds, 0, 1 \right),$$

Moreover this periodic orbit is always stable.

Moreover we shall show that the tools used for proving Theorems 1 to 5 do not provide more periodic solutions outside the ones provided in the mentioned theorems, see Proposition 6.

2. Proof of the results

The tool for proving our results will be the averaging theory. This theory applies to periodic non–autonomous differential systems depending on a small parameter ε. Since the Vallis system already is a T–periodic non–autonomous differential system, in order to apply to it the averaging theory described in section 3 we need to introduce in such system a small parameter. This is reached doing convenient rescalings in the variables (x, y, z), in the parameters (a, b) and in the function $u(t)$. Playing with different rescalings we shall obtain different result on the periodic solutions of the Vallis system. More precisely, in order to study the periodic solutions of the differential system (1), we start doing a rescaling of the variables (x, y, z), of the function $u(t)$, and of the parameters a and b, as follows

$$(2) \qquad \begin{aligned} x &= \varepsilon^{m_1} X, & y &= \varepsilon^{m_2} Y, & z &= \varepsilon^{m_3} Z, \\ u(t) &= \varepsilon^{n_1} U(t), & a &= \varepsilon^{n_2} A, & b &= \varepsilon^{n_3} B, \end{aligned}$$

where ε always is positive and sufficiently small, and m_i and n_j are non–negative integers, for all $i, j = 1, 2, 3$. Then in the new variables (X, Y, Z) system (1) writes

$$(3) \qquad \begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2+m_3} XZ, \\ \frac{dZ}{dt} &= -Z - \varepsilon^{m_1+m_2-m_3} XY + \varepsilon^{-m_3}. \end{aligned}$$

Consequently, in order to have non–negative powers of ε we must impose the conditions

$$(4) \qquad m_3 = 0 \quad \text{and} \quad 0 \leq m_2 \leq m_1 \leq L,$$

where $L = \min\{m_2 + n_3, n_1 + n_2\}$. So system (3) becomes

$$
\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
(5) \quad \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2} XZ, \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_1+m_2} XY.
\end{aligned}
$$

Our aim is to find periodic solutions of system (5) for some special values of m_i, n_j, $i, j = 1, 2, 3$, and after we go back through the rescaling (2) to guarantee the existence of periodic solutions in system (1). In what follows we consider the case where n_2 and n_3 are positives and $m_2 = m_1 < n_1 + n_2$. These conditions lead to the proofs of Theorems 1, 2 and 3. For this reason we present these proofs together in order to avoid repetitive arguments. Moreover, in what follows we consider

$$
K = \int_0^T U(s)ds.
$$

Proofs of Theorems 1, 2 *and* 3: We start considering system (5) with n_2 and n_3 positive and $m_2 = m_1 < n_1 + n_2$. So we have

$$
\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
(6) \quad \frac{dY}{dt} &= -Y + XZ, \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{2m_1} XY.
\end{aligned}
$$

Now we apply the averaging method to the differential system (6). Using the notation of section 3 we have $\mathbf{X} = (X, Y, Z)^T$ and

$$
(7) \qquad\qquad F_0(t, \mathbf{X}) = \begin{pmatrix} 0 \\ -Y + XZ \\ 1 - Z \end{pmatrix}.
$$

We start considering the system

$$
(8) \qquad\qquad \dot{\mathbf{X}} = F_0(t, \mathbf{X}).
$$

Its solution $\mathbf{X}(t, \mathbf{Z}, 0) = (X(t), Y(t), Z(t))$ such that $\mathbf{X}(0, \mathbf{Z}, 0) = \mathbf{Z} = (X_0, Y_0, Z_0)$ is

$$
\begin{aligned}
X(t) &= X_0, \\
Y(t) &= (1 - e^{-t}(1+t))X_0 + e^{-t}Y_0 + e^{-t}tX_0Z_0, \\
Z(t) &= 1 - e^{-t} + e^{-t}Z_0.
\end{aligned}
$$

In order that $\mathbf{X}(t, \mathbf{Z}, 0)$ is a periodic solution we must choose $Y_0 = X_0$ and $Z_0 = 1$. This implies that for every point of the straight line $X = Y$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. Here and in what follows $\mathbb{S}^1$ is the interval $[0, T]$ identifying T with 0.

Observe that, using the notation of section 3, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (X_0, 1)$, and consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(X_0, X_0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$. The fundamental matrix $M_{\mathbf{Z}}(t)$ of (8), satisfying that $M_{\mathbf{Z}}(0)$ is the identity of $\mathbb{R}^3$, is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - \cosh t + \sinh t & e^{-t} & e^{-t}tX_0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

and its inverse matrix $M_{\mathbf{Z}}^{-1}(t)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - e^t & e^t & -e^t tX_0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Since the matrix $M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & e^T T X_0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$ because $T \neq 0$, we can apply the averaging theory described in section 3.

Let F be the vector field of system (6) minus F_0 given in (7). Then the components of the function $M_{\mathbf{Z}}^{-1}(t)F(t, \mathbf{X}(t, \mathbf{Z}, 0))$ are

$$\begin{aligned} g_1(X_0, t) &= -\varepsilon^{n_2} A X_0 + \varepsilon^{n_3} B X_0 + \varepsilon^{-m_1 + n_1 + n_2} A U(t), \\ g_2(X_0, t) &= \varepsilon^{2m_1} e^t t X_0^3 + (1 - e^t) g_1(X_0, t), \\ g_3(X_0, t) &= -\varepsilon^{2m_1} e^t X_0^2. \end{aligned}$$

In order to apply averaging theory of first order we need to consider only terms up to order ε. Analysing the expressions of g_1, g_2 and g_3 we note that these terms depend on the values of m_1 and n_j, for each $j = 1, 2, 3$. In fact, we just need to study the integral of g_1 because $k = 1$. Moreover studying the function g_1 we observe that the only possibility to obtain an isolated zero of the function

$$f_1(X_0) = \int_0^T g_1(X_0, t)dt$$

is assuming that $n_1 + n_2 - m_1 = 1$. Otherwise, the only solution of $f_1(X_0) = 0$ is $X_0 = 0$ which correspond to the equilibrium point $(X_0, Y_0, Z_0) = (0, 0, 1)$ of system (8). The same occurs if n_2 and n_3 are greater than 1 simultaneously. This analysis reduces the existence of possible periodic solutions to the following cases:

(p_1) $n_2 = 1$ and $n_3 = 1$;
(p_2) $n_2 > 1$ and $n_3 = 1$;
(p_3) $n_2 = 1$ and $n_3 > 1$.

In the case (p_1) we have $M_{\mathbf{Z}}^{-1}(t)F_1(t, X(t, \mathbf{Z}, 0)) = -AX_0 + BX_0 + AU(t)$, and then

$$f_1(X_0) = (-A + B)TX_0 + AK.$$

Consequently, if $A \neq B$, then $f_1(X_0) = 0$ implies

$$X_0 = \frac{AK}{T(A - B)}.$$

So, by Theorem 7, system (6) has a periodic solution $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon))$ such that

$$(X(0, \varepsilon), Y(0, \varepsilon), Z(0, \varepsilon)) \to (X_0, Y_0, Z_0) = \left(\frac{AK}{T(A - B)}, \frac{AK}{T(A - B)}, 1 \right)$$

when $\varepsilon \to 0$. Note that the point (X_0, Y_0, Z_0) is an equilibrium point of system (6).

Then, if we take $n_1 = n_2 = n_3 = 1$ and going back through the rescaling (2) of the variables and parameters, we obtain that the periodic solution of system (6) becomes the periodic solution $(x(t), y(t), z(t))$ of system (1) satisfying that

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a - b)}, \frac{aI}{T(a - b)}, 1 \right).$$

Indeed, we observe that

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(a\varepsilon^{-1})(I\varepsilon^{-1})}{T\varepsilon^{-1}(a - b)} = \frac{aI}{T(a - b)}.$$

Moreover, we note that $f_1'(x_0) = \varepsilon f_1'(X_0) = -a + b \neq 0$, so the periodic orbit corresponding to x_0 is stable if $a > b$, and unstable otherwise. So this completes the proof of Theorem 1.

Analogously the function f_1 in the cases (p_2) and (p_3) is

$$f_1(X_0) = TBX_0 + AI\varepsilon^{-n_1} \qquad \text{and} \qquad f_1(X_0) = -TAX_0 + AI\varepsilon^{-n_1},$$

respectively. In the first case the condition $f_1(X_0) = 0$ implies

$$X_0 = -\frac{AK}{TB}.$$

Now we observe that we have $n_2 > 1$ and $n_3 = 1$. So, going back through the rescaling we obtain

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(-a\varepsilon^{-n_2})(I\varepsilon^{-n_1})}{Tb\varepsilon^{-1}} = -\frac{aI}{Tb\varepsilon^{n_1+n_2-2}}.$$

and consequently, choosing $n_1 = 0$ and $n_2 = 2$, we get $x_0 = -aI/(Tb)$. Note also that $f_1'(x_0) = Tb > 0$, then the periodic orbit corresponding to x_0 is always unstable. Thus Theorem 2 is proved.

Finally, in the case (p_3), $f_1(X_0) = 0$ implies $X_0 = K/T$. So, taking $n_1 = 1$ and going back through the rescaling, we have $x_0 = \varepsilon X_0 = \varepsilon I/(T\varepsilon) = I/T$. Additionally, we have that $f_1'(x_0) = -Ta < 0$. Therefore the periodic solution that comes from x_0 is always stable. This proves Theorem 3. $\square$

Proof of Theorem 4: As in the proofs of Theorems 1, 2 and 3 we start considering a more general case in the powers of ε in (5) taking $n_2 > 0$ and $m_2 < m_1 < L$. In this case the function $F_0(t, X)$ of system (24) is

$$(9) \qquad\qquad F_0(t, \mathbf{X}) = \begin{pmatrix} 0 \\ Y \\ 1 - Z \end{pmatrix}.$$

Then the solution $\mathbf{X}(t, \mathbf{Z}, 0)$ of system (24) satisfying $\mathbf{X}(0, \mathbf{Z}, 0) = \mathbf{Z}$ is

$$(X(t), Y(t), Z(t)) = (X_0, e^{-t}Y_0, 1 - e^{-t} + e^{-t}Z_0).$$

This solution is periodic if $Y_0 = 0$ and $Z_0 = 1$. Then for every point of the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. We observe that using the notation of section 3 we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(\alpha) = (0, 1)$. Consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$.

The fundamental matrix $M_{\mathbf{Z}}(t)$ of (24) with F_0 given by (9) satisfying $M_{\mathbf{Z}}(0) = Id_3$ and its inverse $M_{\mathbf{Z}}^{-1}(t)$ are given by

$$M_{\mathbf{Z}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad M_{\mathbf{Z}}^{-1}(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Since the matrix $M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}T$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$, we can apply the averaging theory described in section 3. Again using the notations introduced in the proofs of Theorems 1, 2 and 3, since $k = 1$ we will look only to the integral of the first coordinate of $\mathcal{F} = (f_1, f_2, f_3)$. In this case we have

$$g_1(X_0, Y_0, Z_0, t) = -\varepsilon^{n_2} A X_0 + \varepsilon^{-m_1 + n_1 + n_2} A U(t).$$

Comparing this function g_1 with the same function obtained in the proof of Theorems 1, 2 and 3, it is easy to see that this case correspond to the case (p_3) of the mentioned theorems. Then, in order to have periodic solutions, we need to choose $n_2 = 1$ and $n_1 + n_2 - m_1 = 1$. So, following the steps of the proof of case (p_3) by choosing $n_1 = 1$ and coming back through the rescaling (2) to system (1), Theorem 4 is proved. $\qquad\square$

Proof of Theorem 5: We start considering system (5) with $n_3 = 2$, $n_2 > 0$, $m_1 = n_1 + n_2$ and $m_2 < m_1 < m_2 + n_3$. With these conditions system (5) becomes

$$\frac{dX}{dt} = -\varepsilon^{n_2} A X + \varepsilon^{m_2 - n_1 - n_2 + n_3} B Y + A U(t),$$

(10)
$$\frac{dY}{dt} = -Y + \varepsilon^{-m_2 + n_1 + n_2} X Z,$$

$$\frac{dZ}{dt} = 1 - Z - \varepsilon^{m_2 + n_1 + n_2} X Y.$$

Again we will use the averaging theory described in section 3. So considering $\mathbf{X} = (X, Y, Z)^T$ we obtain

(11)
$$F_0(t, \mathbf{X}) = \begin{pmatrix} A U(t) \\ -Y \\ 1 - Z \end{pmatrix}.$$

Now we note that the solution $\mathbf{X}(t, \mathbf{Z}, 0) = (X(t), Y(t), Z(t))$ such that $\mathbf{X}(0, \mathbf{Z}, 0) = \mathbf{Z} = (X_0, Y_0, Z_0)$ of the system

(12)
$$\dot{\mathbf{X}} = F_0(t, \mathbf{X})$$

is

$$X(t) = X_0 + \int_0^t A U(s) ds, \quad Y(t) = e^{-t} Y_0, \quad Z(t) = 1 - e^{-t} + e^{-t} Z_0.$$

Since $I = 0$ and $J(t) \neq 0$ for $0 < t < T$, in order that $\mathbf{X}(t, \mathbf{Z}, 0)$ is a periodic solution we need to fix $Y_0 = 0$ and $Z_0 = 1$. This implies that for every point in a neighbourhood of X_0 in the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$.

Following the notation of section 3, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (0, 1)$. Hence $\mathcal{M}$ is an one–dimensional manifold $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$ and the fundamental matrix $M_{\mathbf{Z}}(t)$ of (12) satisfying that $M_{\mathbf{Z}}(0)$ is the identity of $\mathbb{R}^3$ is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & e^{-t}
\end{pmatrix}.
$$

It is easy to see that the matrix $M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$
\Delta = \begin{pmatrix}
1 - e^T & 0 \\
0 & 1 - e^T
\end{pmatrix},
$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$. Then the hypotheses of Theorem 7 are satisfied. Now the components of the function $M_{\mathbf{Z}}^{-1}(t) F(t, X(t, \mathbf{Z}, \mathbf{0}))$ are

$$
g_1(X_0, t) = -\varepsilon^{n_2} A \left(X_0 + \int_0^t AU(s)ds \right) + AU(t),
$$

$$
g_2(X_0, t) = \varepsilon^{-m_2 + n_1 + n_2} \left(X_0 + \int_0^t AU(s)ds \right) e^t,
$$

$$
g_3(X_0, t) = 0.
$$

Taking n_1 and n_2 equal to one and observing that $k = 1$ and $n = 3$, we are interested only in the first component of the function $F_1 = (F_{11}, F_{12}, F_{13})$ described in section 3. Indeed, applying the averaging theory we must study the zeros of the first component of the function

$$
\mathcal{F}(X_0) = (f_1(X_0), f_2(X_0), f_3(X_0)) = \int_0^T M_{\mathbf{Z}}^{-1}(t, \mathbf{Z}) F_{11}(t, \mathbf{X}(t, \mathbf{Z})) dt.
$$

Since

$$
F_{11} = -A \left(X_0 + \int_0^t AU(s)ds \right),
$$

then

$$
\begin{aligned}
f_1(X_0) &= \int_0^T -A \left(X_0 + \int_0^t AU(s)ds \right) dt \\
&= -ATX_0 - A^2 \int_0^T \left(\int_0^t U(s)ds \right) ds.
\end{aligned}
$$

Therefore, from $f_1(X_0) = 0$ we obtain

$$X_0 = -\frac{A}{T} \int_0^T \left(\int_0^t U(s)ds \right) ds \neq 0.$$

So, using rescaling (2) we get

$$x_0 = \varepsilon^2 X_0 = -\varepsilon^2 \frac{a\varepsilon^{-1}}{\varepsilon T} \int_0^T J(s)ds = -\frac{a}{T} \int_0^T J(s)ds.$$

Moreover, since $f_1'(x_0) = -a/T < 0$, because a and ε are positive, the $T-$periodic orbit detected by the averaging theory is always stable. This ends the proof. □

The following shows that there is no other configuration of the powers of ε defining the rescaling (2) for which we can detect other periodic solutions of system (1) using the averaging theory.

Proposition 6. *By using the averaging theory through the rescalings* (2) *no other periodic solutions, except the ones presented in Theorems* 1, 2, 3, 4 *and* 5, *can be found in the Vallis system* (1).

In order to prove Proposition 6 we will study all possible powers of the different $\varepsilon's$ in system (5). Indeed, we consider the set $P = \{n_2, -m_1 + m_2 + n_3, -m_1 + n_1 + n_2, m_1 - m_2\}$ of the relevant powers of ε in this system (see (4) and (5)), and observe that each integer of P must be non–negative. Therefore, we will study each one of the 16 possible combinations of values of the elements of P taking into account conditions (4). We start considering $n_2 > 0$. Then we have the following eight cases:

Case 1: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 = n_1 + n_2$,
Case 2: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 < n_1 + n_2$,
Case 3: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$,
Case 4: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,
Case 5: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$,
Case 6: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,
Case 7: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 = n_1 + n_2$,
Case 8: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 < n_1 + n_2$.

The remainder cases from 9 to 16 are the same than the cases from 1 to 8, respectively, taking $n_2 = 0$ instead of $n_2 > 0$.

We observe that the case 4 was studied in Theorems 1, 2 and 3 Additionally, Theorem 4 concerns to case 8, and Theorem 5 deals to case 7. Thus we will eliminate these cases in the proof of Proposition 6.

In the other cases we will prove that some hypotheses of the averaging method presented in section 3 do not hold.

Proof of Proposition 6: First we prove the proposition using system (5) in case 2. Indeed, considering $\mathbf{X} = (X, Y, Z)$, in case 2 system (24) becomes

$$(13) \qquad \dot{\mathbf{X}} = F_0(t, \mathbf{X}) = (BY, -Y + XZ, 1 - Z)^T.$$

This last differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. It is easy to see that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic. Now substituting the solution $Z(t)$ in the second differential equation of (13) and solving the system of differential equations $\dot{X} = BY$, $\dot{Y} = -Y + X$ we get

$$X(t) = \frac{1}{2C} \left(C_1 e^{\frac{1}{2}(-C-1)t} + C_2 e^{\frac{1}{2}(C-1)t} \right),$$

$$Y(t) = \frac{1}{2C} \left(C_3 e^{\frac{1}{2}(-C-1)t} + C_4 e^{\frac{1}{2}(C-1)t} \right),$$

where $C = \sqrt{1 + 4B} > 1$, $C_1 = (C-1)X_0 - 2BY_0$, $C_2 = (C+1)X_0 + 2BY_0$, $C_3 = -2X_0 + (C+1)Y_0$ and $C_4 = 2X_0 + (C-1)Y_0$.

Without loss of generality we will study the conditions that turn the solution $X(t)$ into a periodic function. In order to do this, we need to choose C_1 and C_2 equal to zero because $C > 1$. Fixing $C_1 = 0$ we obtain

$$X_0 = \frac{-2BY_0}{C - 1}.$$

Replacing this value into C_2 we get $(1 + 4B + C)Y_0$ which is positive unless $Y_0 = 0$. On the other hand the value $Y_0 = 0$ implies $X_0 = 0$, and since $Z_0 = 1$ we have the equilibrium point $(0, 0, 1)$ of system (13). This implies that system (13) has no periodic solutions, and then the averaging method described in section 3 cannot be applied in this case. Moreover, there is no loss of generality when we study only the solution $X(t)$ because if one of the functions $X(t)$, $Y(t)$ or $Z(t)$ of (24) is not periodic, system (13) cannot have a periodic solution. We will use this fact to end the proof of Proposition 6 in some other cases below.

In what follows we prove Proposition 6 for system (5) in case 10. Indeed observe that system (24) now writes

$$(14) \qquad \dot{\mathbf{X}} = (-AX + BY, -Y + XZ, 1 - Z)^T.$$

As before we take the solution $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ of $\dot{Z} = 1 - Z$ and we replace this solution with $Z_0 = 1$ in the other differential equations

of (14). Therefore the solutions $X(t)$ is

$$X(t) = \frac{1}{2D}\left(D_1 e^{\frac{1}{2}(-A-1-D)t} + C_2 e^{\frac{1}{2}(-A-1+D)t}\right)$$

where $D = \sqrt{(A-1)^2 + 4B} > 0$, $D_1 = (A - 1 + D)X_0 - 2BY_0$ and $D_2 = (-A + 1 + D)X_0 + 2BY_0$. We note that this expression is very similar to the expression of the solution $X(t)$ of system (5) in case 2 just taking A as zero. Moreover, it is possible to show that the same arguments used in case 2 are also true in this case, and consequently the averaging method does not apply to system (5) in case 10.

In case 12 we have

(15) $$\dot{\mathbf{x}} = (-AX, -Y + XZ, 1 - Z)^T.$$

So the solutions $X(t)$ and $Z(t)$ for this system are, respectively, $X(t) = e^{-At}X_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. Then choosing $X_0 = 0$ and $Z_0 = 1$ in order that $X(t)$ and $Z(t)$ be periodic, the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0$, and consequently we have to take $Y_0 = 0$. However, the point $(X_0, Y_0, Z_0) = (0, 0, 1)$ is the equilibrium point of system (15), and therefore system (15) has no periodic solutions. Thus, in case 12 again we cannot apply the averaging theory.

In case 14 system (24) is

(16) $$\dot{\mathbf{X}} = (-AX + BY, -Y, 1 - Z)^T,$$

whose solutions $Y(t)$ and $Z(t)$ starting at Y_0 and Z_0 are, respectively, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. These solutions are periodic if $Y_0 = 0$ and $Z_0 = 1$, and with these values the solution $X(t)$ writes $X(t) = e^{-At}X_0$. So, since $A \neq 0$, we need to take $X_0 = 0$ for having $X(t)$ periodic. The conclusion of Proposition 6 in this case follows as in case 12.

For proving Proposition 6 in case 16 we observe that the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) given by

(17) $$\dot{\mathbf{X}} = (-AX, -Y, 1 - Z)^T,$$

are $X(t) = e^{-At}X_0$, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. The values X_0, Y_0 and Z_0 for which these solutions are periodic are $(X_0, Y_0, Z_0) = (0, 0, 1)$. So as before we cannot apply the averaging theory.

Now we prove that the averaging method does not work in system (5) in case 5. In fact, in this case the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) starting at (X_0, Y_0, Z_0) are

$$(X(t), Y(t), Z(t)) = \left(X_0 + \int_0^t AU(s)ds, e^{-t}Y_0, 1 - e^{-t} + e^{-t}Z_0\right),$$

where now we suppose that $\int_0^T u(s)ds = 0$ in order that $x(t)$ be a $T-$periodic solution. Looking to the expressions of $Y(t)$ and $Z(t)$, it is easy to see that $Y_0 = 0$ and $Z_0 = 1$ are the only values of Y_0 and Z_0 for which $Y(t)$ and $Z(t)$ are periodic. We observe that using the notation of section 3, we have $k = 1$, $n = 3$ and the fundamental matrix $M_{\mathbf{Z}}(t)$ is

$$\begin{pmatrix} 1 & B(1 - \cosh(t) + \sinh(t)) & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

and its inverse matrix is

$$M_{\mathbf{Z}}^{-1}(t) = \begin{pmatrix} 1 & B(1 - e^t) & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

So the matrix $M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}T$ does not have a 1×2 zero matrix in the upper right corner, because

$$M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}T = \begin{pmatrix} 0 & B(e^T - 1) & 0 \\ 0 & 1 - e^T & 0 \\ 0 & 0 & 1 - e^T \end{pmatrix}.$$

Then, since B is positive, we cannot apply averaging method in case 5.

Case 6 is similar to case 5. In fact, the solution $X(t)$, $Y(t)$ and $Z(t)$ of system (24) starting at (X_0, Y_0, Z_0) eliminating the non–periodic terms is

$$(X(t), Y(t), Z(t)) = (X_0, 0, 1),$$

and following the steps of section 3 we obtain the same matrix $M_{\mathbf{Z}}^{-1}(t)$ of case 5. Hence we cannot apply the averaging method in this case.

Next we prove Proposition 6 in case 3. The solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) are

$$X(t) = X_0 + \int_0^t AU(s)ds,$$

$$Y(t) = X_0 - e^{-t}X_0 + e^{-t}Y_0 + \int_0^t AU(s)ds - e^{-t}\int_0^t Ae^s U(s)ds,$$

$$Z(t) = 1 - e^{-t} + e^{-t}Z_0,$$

where we suppose that $I = 0$ in order that $X(t)$ be a $T-$periodic solution. Observe that if $I \neq 0$, $X(t)$ is not periodic and then we cannot apply the averaging method. Indeed the expression of $Z(t)$ implies that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic.

Moreover, we take $X_0 = Y_0 + W$ such that the solutions $X(t)$, $Y(t)$ and $Z(t)$ become

$$X(t) = Y_0 + W + A \int_0^t U(s)ds,$$

$$Y(t) = Y_0 + W + A \int_0^t U(s)ds - e^{-t}\left(W - A \int_0^t e^s U(s)ds\right),$$

$$Z(t) = 1,$$

where $\int_0^t U(s)ds$ is periodic because $I = 0$ and we suppose that $e^{-t}(W - A \int_0^t e^s U(s)ds)$ is periodic. Note that if a such W does not exits, then $Y(t)$ is no–periodic and the averaging theory does not apply. Hence, we assume that a such W exits and the solution $Y(t)$ is $T-$periodic.

We note that if $U(t) = \cos t$ and $W = A/2$ the solutions $X(t)$, $Y(t)$ and $Z(t)$ are periodic, because with these considerations we have

$$X(t) = Y_0 + (A/2) + A \sin t,$$
$$Y(t) = (1/2)(A + 2Y_0 - A \cos t + A \sin t),$$
$$Z(t) = 1,$$

which are periodic functions. However we want to consider the general case instead of this particular case $U(t) = \cos t$ and $W = A/2$. Hence using the notation of section 3, we have $k = 1$, $n = 3$ and the fundamental matrix $M_{\mathbf{Z}}(t)$ is

$$\begin{pmatrix} e^t & 1 - e^t & e^t E(A, W, Y_0, t) \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

where $E(A, W, Y_0, t) = \int_0^t e^{-2s}(Y_0 + W + A \int_0^s U(w)dw)ds$. Its inverse matrix $M_{\mathbf{Z}}^{-1}(t)$ is

$$\begin{pmatrix} e^{-t} & 1 - \cosh t + \sinh t & -e^t E(A, W, Y_0, t) \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Then the matrix $M_{\mathbf{Z}}^{-1}(0) - M_{\mathbf{Z}}^{-1}T$ has a 2×2 lower right matrix

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

whose determinant is zero. Then we cannot apply the averaging method in case 3.

We study system (5) in case 1. Now system (24) is

$$(18) \qquad (\dot{X}, \dot{Y}, \dot{Z}) = (BY + AU(t), -Y + XZ, 1 - Z)^T.$$

This differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. As before if $Z_0 = 1$ then $Z(t)$ is a periodic solution. Now substituting the solution $Z(t)$ in the second differential equation of (18) with $Z_0 = 1$ and solving the system of differential equations $\dot{X} = BY + AU(t)$, $\dot{Y} = -Y + X$ we get solutions very similar to the ones obtained in case 2. In fact, denoting by $X_2(t)$ and $Y_2(t)$ the solutions of case 2, for case 1 the solutions $X(t)$ can be written as

$$X(t) = X_2(t) + (1/2C)g_2(A, B, t)e^{\frac{1}{2}(-C-1)t},$$

where g_2 is

$$\frac{A}{2C}(-1 + C + Ce^{Ct}) \int_0^t e^{-\frac{1}{2}(-1+C)s} \left(1 + C + (-1 + C)e^{Cs}\right) U(s)ds +$$

$$\frac{2AB}{C}(1 - e^{Ct}) \int_0^t e^{-\frac{1}{2}(-1+C)s} \left(-1 + e^{Cs}\right) U(s)ds.$$

We observe that g_2 does not depend neither of X_0 nor of Y_0. For this reason we cannot eliminate the non–periodic terms of $X_2(t)$ through the expression $(1/2C)g_2(A, B, t)e^{\frac{1}{2}(-C-1)t}$, whatever be the function $g_2(A, B, t)$ chosen. So as we see in case 2 we must choose $(X_0, Y_0) = (0, 0)$ in order that $X_2(t)$ be periodic. Since $Z_0 = 1$, system (18) has no submanifold of periodic solutions as needs the averaging theory described in section 3.

Case 9 is similar to case 1 in the sense that there is no choice of X_0, Y_0 and Z_0 in such way that the solution of the system

$$(19) \qquad \dot{\mathbf{X}} = (-AX + BY + AU(t), -Y + XZ, 1 - Z)^T.$$

corresponding to system (5) in case 9 has a submanifold of periodic solutions. As before, $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ is the solution of the last differential equation of system (19), and the value Z_0 for which this solution is periodic is $Z_0 = 1$. Substituting this solution into system (19) and solving it, we obtain a solution similar to the solution of system (14) in case 10 denoted here by $X_{10}(t)$. We have

$$X(t) = X_{10}(t) + g_{10}(A, B, t),$$

where g_{10} is

$$\frac{A}{2D}(-1 + A\left(1 - e^{Dt}\right) + D + (1 + D)e^{Dt}) \int_0^t e^{\frac{1}{2}(1+A-D)s}\left(1 + D+\right.$$

$$\left.(-1 + D)e^{Ds} + A\left(-1 + e^{Ds}\right)\right)U(s)ds + \frac{2AB}{D}(1 - e^{Dt})$$

$$\int_0^t e^{\frac{1}{2}(1+A-D)s}\left(-1 + e^{Ds}\right)U(s)ds.$$

Note that g_{10} does not depend neither of X_0 nor of Y_0. The conclusion of this case follows from the fact that X_{10} is non–periodic unless $(X_0, Y_0) = (0, 0)$, and using the same arguments of the proof of case 1.

Now we consider system (5) in case 11

$$(20) \qquad \dot{\mathbf{x}} = (-AX + AU(t), -Y + XZ, 1 - Z)^T.$$

Considering $A \neq 1$, as before we have $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ and choose $Z_0 = 1$ because $Z(t)$ must be periodic. The solution $X(t)$ is

$$X(t) = e^{-At}X_0 + A\int_0^t e^{As}U(s)ds.$$

This means that we must take $X_0 = 0$ to have $X(t)$ periodic. Substituting $X_0 = 0$ and $Z(t) = 1$ in $Y(t)$, it becomes

$$Y(t) = Y_0 e^{-t} + \frac{1}{A-1}e^{-(A+1)t}h_{11}(A, t),$$

where now h_{11} does not depends on Y_0 and writes

$$A(e^{At} - e^t)\int_0^t e^{As}U(s)ds + Ae^{At}\int_0^t \left(e^s - e^{As}\right)U(s)ds.$$

Since $A \neq 1$ and h_{11} does not depend on Y_0 we cannot eliminate the non–periodic term $Y_0 e^{-t}$ of $Y(t)$ unless we take $Y_0 = 0$. Consequently, as in cases 1 and 9 the averaging theory does not work in case 11.

Moreover, if $A = 1$, we have the same solutions $X(t)$ and $Z(t)$. So, considering again $X_0 = 0$ and $Z_0 = 1$ the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0 + h(t)$, where h does not depend on Y_0. Hence, considering $A = 1$ again we cannot eliminate the non–periodic term $Y_0 e^{-t}$ of $Y(t)$ unless $Y_0 = 0$, and therefore the averaging cannot be applied.

Cases 13 and 15 can be proved in a similar way. More precisely, systems (24) corresponding to system (5) in cases 13 and 15 are

$$(21) \qquad \dot{\mathbf{X}} = (-AX + BY + AU(t), -Y, 1 - Z)^T,$$

and

$$(22) \qquad \dot{\mathbf{X}} = (-AX + AU(t), -Y, 1 - Z)^T,$$

respectively. In both cases solutions $Y(t)$ and $Z(t)$ are $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. So in order to be $Y(t)$ and $Z(t)$ periodic we take $Y_0 = 0$ and $Z_0 = 1$. Then the solution $X(t)$ becomes

$$X(t) = e^{-At}X_0 + A \int_0^t e^s U(s)ds.$$

Again once $g(t) = \int_0^t e^s U(s)ds$ does not depend on X_0 it is not possible to eliminate the non–periodic term $e^{-At}X_0$ from $X(t)$ unless we take $X_0 = 0$. Therefore both systems (21) and (22) do not have a submanifold $\mathcal{M}$ filled with periodic solutions. Hence the averaging theory cannot be applied in cases 13 and 15. $\square$

3. The Averaging Theory for Periodic Orbits

Now we present the basic results on the averaging theory of first order that we need for proving our results.

Consider the problem of bifurcation of T–periodic solutions from the differential systems of the form

$$(23) \qquad \dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon),$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$(24) \qquad \dot{\mathbf{x}} = F_0(t, \mathbf{x}),$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Indeed let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (24) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$(25) \qquad \dot{\mathbf{y}} = D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}.$$

Then we have the following result.

Theorem 7. *Assume there exists a k–dimensional submanifold $\mathcal{M}$ filled with T–periodic solutions of system (24). Let V be an open and bounded subset of $\mathbb{R}^k$ and let $\beta : Cl(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{M} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)); \alpha \in Cl(V)\}$ and that for each $\mathbf{z}_\alpha \in \mathcal{M}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (24) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{M}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}$ of (25) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det \Delta_\alpha \neq 0$.*

Define $\mathcal{F} : Cl(V) \to \mathbb{R}^k$ as

$$\mathcal{F}(\alpha) = \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t, \mathbf{z}_\alpha) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt.$$

Then the following statements hold.

(a) *If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((\partial \mathcal{F}/\partial \alpha)(a)) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (23) such that $\mathbf{x}(t, \varepsilon) \to \mathbf{z}_\alpha$ when $\varepsilon \to 0$.*

(b) *The type of stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $((\partial \mathcal{F}/\partial \alpha)(a))$.*

In fact, the result of Theorem 7 is a classical result due to Malkin [6] and Roseau [7]. For a shorter proof of Theorem 7, item (a), see [1].

For additional information on averaging theory see the book [8].

Acknowledgments

References

[1] A. Buică, J. P. Françoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Comm. on Pure and Appl. Anal. **6** (2007), 103–111.

[2] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Revised and corrected reprint of the 1983 original*, Applied Mathematical Sciences **42**, Springer-Verlag, New York, 1990.

[3] A. Kanatnikov and A. Krishchenko, *Localization of invariant compact sets of nonautonomous systems*, Differ. Equ. **45**, No. 1, (2009), 46–52.

[4] A. P. Krishchenko and K. E. Starkov, *Localization of Compact Invariant Sets of Nonlinear Time-Varying Systems*, Intern. Journal of Bifurcation and Chaos , (2008), no. 5, pp. 1599–1604.

[5] J. Llibre and C. Vidal, *Periodic solutions of a periodic FitzHugh-Nagumo differential system*, to appear.

[6] I. G. Malkin, *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956.

[7] M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol. 8 Springer–Verlag, Berlin–New York, 1966.

[8] J. A. Sanders, F. Verhulst and J. Murdock, *Averaging method in non-linear dynamical systems*, Appl. Math. Sci., vol. **59**, Springer, New York, 2007.

[9] D. Strozzi, *On the origin of interannual and irregular behaviour in the El Niño properties*, (1999), Report of Department of Physics, Princeton University, available at the WEB.

[10] G. K. Vallis, *Conceptual models of El Niño and the southern oscillation*, J. Geophys. Res. **93** (1988), 13979–13991.

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