POINCARÉ RECURRENCE THEOREM FOR NON-SMOOTH VECTOR FIELDS

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Abstract. In this paper some ergodic aspects of non-smooth vector fields are studied. More specifically, the concepts of recurrence and invariance of a measure by a flow are discussed and two versions of the classical Poincaré Recurrence Theorem are presented. The results allow us to soften the hypothesis of the classical Poincaré Recurrence Theorem by admitting non-smooth multi-valued flows. The methods used in order to prove the results involve elements from both measure theory and topology.

1. Introduction

The study of the so-called non-smooth vector fields (NSVF), also addressed in the literature as discontinuous or piecewise smooth vector fields, has received special attention from the mathematical community in the last years, mainly due to the closeness of such area to applied sciences as mechanics, engineering, electronic and biology, as well as social and financial sciences (for applications of NSVF see, for instance, the book of di Bernardo, Budd, Champneys and Kowalczyk – [6]).

Roughly speaking, in the theory of NSVF is admitted the existence of one or more codimension one manifolds separating the space into two or more regions where are defined distinct vector fields.

It means, among others facts, that on the boundary of each region we have defined at least two vector fields, so the trajectories passing through such *switching* regions may be non-regular or even non-unique, depending on the intrinsic geometry of the vector fields and the switching manifold. Some papers dealing with more than two vector fields which contain a richer dynamics can be found in [9, 12, 20], for instance. Moreover, as we will see, it is also possible for a particular trajectory to be confined onto the switching manifold itself (more details concerning NSVF will be provided timely in this paper).

The terminology which introduces the behavior of the trajectories on the switching manifold, also called *discontinuity manifold*, was presented by Filippov in 1988 (see [11]). Before that, the theory of NSVF was underdeveloped and it had a strong relation with manifolds with boundaries. In fact, one of the forerunner works is a paper of Teixeira in 1977 (see [18]). More recently, many other authors have contributed to the progress of the NSVF's theory. Some landmarks we can cite are the works [1], [13], [19] and references therein.

One of the most important goals concerning the theory of NSVF is to look over the validity of the results coming from the classical theory of dynamical systems into

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the non-smooth scenario. By classical we mean not only topological but also, for instance, ergodic, symbolic and discrete points of view. Though, in this direction, some particular results have been obtained through the last years. For instance, it is pretty clear that the *Existence and Uniqueness Theorem* is not true in the non-smooth context, as suggested previously. On the other hand, under suitable hypotheses, we already know that *Poincaré Index Theorem*, *Poincaré-Bendixson Theorem* and *Peixoto Theorem* possess versions for NSVF (see [3], [5] and [17]). Moreover, some works have recently addressed different concepts from general dynamics in the NSVF scenario. In such works, the authors introduce, for instance, the concepts of invariance, minimality and chaoticity for NSVF and they achieve surprising features concerning these objects, such as the existence of non-trivial minimal sets and chaotic vector fields in dimension 2 (see for instance [2], [8] and [15]).

We should remark that, in general, presenting a classical result from dynamical systems' theory for the non-classical NSVF involves the process of softening the hypotheses of that result by allowing multivalued flows, which are eventually non-regular. It means that we do not assume neither uniqueness nor smoothness of solutions, which prints in some way a randomic feature in PSVF.

2. Setting the problem

Our goal is to study some ergodic aspects of the NSVF. More specifically, we discuss the validity of one of the most important theorems from ergodic theory, namely, the Poincaré Recurrence Theorem. It basically deals with the recurrence of a set of points depending on invariant measures for the considered NSVF. We prove that under suitable conditions, such theorem also holds for a special class of NSVF; we also show that it does not hold in the complementary of such class.

Nevertheless, it is known that in the classical ergodic theory the Poincaré Recurrence Theorem can be stated in two manners (see, for instance, [21], so in an analogously way, in this work the first manner will be called *measurable version*, which states that "*given a finite invariant measure, the orbit of almost every point of every measurable set E returns to E infinitely many times*". This version will be presented in Theorem 5.1 of Section 5. On the other hand, the other version will be called *topological version*. This version states that "*given a finite invariant measure, almost every point is recurrent by the non-smooth vector field*" and it will be presented in Theorem 5.2 of Section 5. As we commented before, our goal here is to prove these two versions of Poincaré Recurrence Theorem in the scenario of NSVF. In order to do this, since the flow related to a NSVF is not necessarily unique, we must introduce consistent definitions, for instance, on what we understand by a recurrent point and its dependence on the flow passing through such points. Moreover, we shall discuss the meaning of concepts as invariant measures and flows for NSVF as well as other concepts related on.

This paper is organized as follows. In Section 3 we present some basic concepts concerning NSVF, recurrence and invariant measures. In section 4 we make use of some examples to discuss the main hypotheses on what we based our theorems; we also compare the class of systems we are working with to classical dynamical systems. The main results, namely, the Poincaré Recurrence Theorem in the measurable and topological versions, are stated and proved in Section 5. Finally, in

Section 6 we present the conclusions and some brief comments about future works in ergodic theory joint with NSVF.

3. Preliminars

In what follows we introduce the first ideas about non-smooth vector fields following the methodology stated by Filippov in [11]. In order to do that, let M be an open set of $\mathbb{R}^n$ and consider a codimension one manifold Σ of $\mathbb{R}^n$ given by $\Sigma = F^{-1}(0)$, where $F : V \to \mathbb{R}$ is a C^r smooth function having $0 \in \mathbb{R}$ as a regular value. We designate by χ the space of C^r-vector fields on $M \subset \mathbb{R}^n$, with $r \geq 1$ large enough for our purposes and take Ω the space of vector fields $Z : M \to \mathbb{R}^n$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases} \tag{1}$$

where $X, Y \in \chi$. We call Σ the *switching manifold*, which is the region where the non-smooth vector field Z switches from X to Y and viceversa. Consequently Z is bi-valuated on Σ.

Concerning the behavior of the trajectories reaching the switching manifold Σ, consider Lie derivatives

$$X.F(p) = \langle \nabla F(p), X(p) \rangle \quad \text{and} \quad X^i.F(p) = \left\langle \nabla X^{i-1}.F(p), X(p) \right\rangle, i \geq 2,$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^n$. Filippov distinguished three regions on the switching manifold taking into account the geometry of the vector fields defined on Σ, as follows:

(i) $\Sigma^c \subseteq \Sigma$, the *sewing region*, where $(X.f)(Y.F) > 0$;
(ii) $\Sigma^e \subseteq \Sigma$, the *escaping region*, where $(X.F) > 0$ and $(Y.F) < 0$;
(iii) $\Sigma^s \subseteq \Sigma$, the *sliding region*, where $(X.F) < 0$ and $(Y.F) > 0$.

We denote by $\Sigma^t \subseteq \Sigma$ the set of points satisfying $(X.F(q))(Y.F(q)) = 0$, which means the tangency points of the trajectories of X or Y with Σ. If $X^n.F(q) = X.\left(X^{n-1}.F\right)(q) \neq 0$ we say that p is a tangency of order even (respectively odd) if n is even (respectively odd). Moreover, we say that p is a *regular* tangency if there is a trajectory reaching p and a *singular* one otherwise.

Next we present the definition of a flow induced by a NSVF under the methodology presented by Filippov (see [11]). Before that, we remark that a *local flow* (or *local trajectory*) $\sigma(t,p)$ passing through a point p at the time $t = 0$ is any trajectory reaching p defined for a finite or infinite time, so it is not necessarily unique (observe that $\sigma(t,p)$ is unique if $p \in \Sigma^e \cup \Sigma^c$). Observe also that if $Z = (X,Y)$ is a NSVF and $p \in \Sigma^s$, for instance, the local flow by p could be the trajectory coming from the vector fields X or Y or slide on Σ. For more details about local flows see [2] and [14].

Definition 3.1. *A **global flow** $f_t(t,p_0)$ of $Z \in \Omega$ passing through p_0 is a union*

$$f_t(t,p_0) = \bigcup_{i \in \mathbb{Z}} \Gamma_i$$

where, for each $i \in \mathbb{Z}$, $\Gamma_i = \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$ with $\sigma_i(t,p_i)$ being a preserving-orientation local trajectory satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $(t_i)_{i \in \mathbb{Z}}$ is a sequence such that $t_i < t_{i+1}$, $\forall i \in \mathbb{Z}$, with $t_0 = 0$ and $t_i \to \pm\infty$ as

$i \to \pm\infty$. A global flow is ***positive*** *(respectively,* ***negative****) if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

We observe that the definition of global flow of a NSVF is slightly different from the classical definition of flow once may occur no uniqueness of trajectories. Thus we can not assure the uniqueness of local flows and consequently the same hold for global flows. In order to consider every possible choice for the trajectory starting or ending in a set A, we define

$$\varphi_t(A) = \bigcup_{p \in A} f_t(t, p).$$

that we call here the *saturation* of the set A. Of course, analogously we can define the saturation of a point p by taking $A = \{p\}$.

Finally, we stress that although we are unable to guarantee the smoothness of the trajectories of a NSVF, by Definition 3.1 every flow passing through a given point is continuous. Consequently it makes sense to consider a measure which is invariant for a non-smooth flow once every continuous function is clearly a measurable one. Despite of these considerations, now we define the concept of an invariant measure by a flow.

Definition 3.2. *Given a measure μ and a flow f_t of a NSVF $Z \in \Omega$ defined in a compact manifold M, we say that μ is invariant by f_t if*

$$\mu(E) = \mu(f_{-t}(E)),$$

for all measurable subsets $E \subset M$ and for all $t \in \mathbb{R}$.

Remark 3.1. *Although the classical definition of invariant measures takes into account only positive times, in the scenario of NSVF the orientation of the time has an important role (see Definition 1). That is the reason why we consider $t \in \mathbb{R}$ in Definition 3.2 instead of only positive values of t.*

We observe that, if Z is smooth (considering, for instance, $Z = (X, Y)$ with $X = Y$), Definition 3.2 coincides with the classical definition of an invariant measure μ by a flow f_t of Z, as expected. Note that in such scenario the flow f_t of Z is unique by the Existence and Uniqueness Theorem. However, if Z is non-smooth, it is not clear that μ is still invariant if we consider other flow g_t of Z. Consequently, the definition of invariance of a measure μ by a flow f_t strongly depends on the particular flow we have considered.

Nevertheless, if the invariance does not depend on the flow, we will say that the measure is invariant for the non-smooth vector field Z, as stated in the next definition.

Definition 3.3. *We say that a measure μ is invariant by a non-smooth vector field $Z \in \Omega$ if μ is invariant by every flow f_t of Z.*

Another fundamental concept in the ergodic theory we must introduce to present the results is the notion of recurrence, which is very clear in the context of smooth and discrete dynamical systems, but in the non-smooth scenario it could be clarified in order to avoid misunderstanding.

Definition 3.4. *Given a flow $(f_t)_t$ of a NSVF we say that a point $x \in M$ is recurrent by this flow if there exists a sequence $(t_i)_i$, $t_i \to \infty$ when $i \to \infty$, such that $f_{t_i}(x) \to x$ when $i \to \infty$.*

Again, we observe that this definition coincides to the classical definition of recurrence since in the classical case we have uniqueness of the flow (note that this is the case in Definition 3.4 once we have fixed the flow f_t). However, due to the non-uniqueness of the flow of a NSVF, we must also provide a definition of recurrence by a NSVF taking into account these facts. Nevertheless, it is probably simpler to introduce the idea of non-recurrence for NSVF. Indeed, a point q should be non-recurrent if there exists a small neighborhood V_q of q and a finite time $T_q > 0$ such that every flow passing through q does not return to V_q, for any time $t \in \mathbb{R}$ with $|t| > T_q$. It sounds reasonable once we can have infinitely many flows passing though q. Therefore, the definition of recurrence must be the following.

Definition 3.5. *Given a non-smooth vector field $Z \in \Omega$ defined in a compact manifold M, we say that a point $x \in M$ is recurrent by Z if x is recurrent by some flow f_t of Z.*

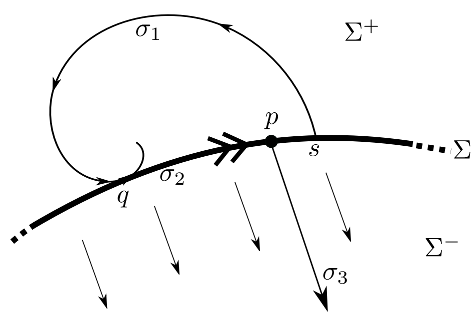

FIGURE 1. The recurrence of a point depends on the flow passing through this point.

Example 3.1. *Consider the non-smooth vector Z which realizes the phase portrait of Figure 1, that is, $X \in \Sigma^+$ has a stable focus and $Y \in \Sigma^-$ is a constant vector field whose orbits point outside Σ. Consider now the following trajectories: (i) σ_1 the arc of trajectory of X connecting the point $s \in \Sigma^e$ to the point $q \in \Sigma^t$; (ii) σ_2 the arc of local flow contained in the trajectory coming from the sliding vector field which starts in the point q and finishes in the point s and (iii) the trajectory of Y starting in a point $p \in \Sigma^e$ with p contained in the interior of the trajectory σ_2. Observe that any point x that belongs to the pseudo-cycle $\Gamma_0 = \sigma_1 \cup \sigma_2$ is recurrent for Z once the trajectory which remains on Γ_0 returns to x infinitely many times, although some trajectories passing through x are not recurrent (for instance, if $x = p$, then x is not recurrent for σ_3). On the other hand, by considering a point $y \in \sigma_3 \setminus \Sigma$, the trajectories by y is unique and does not return to any neighborhood of y for all time $t \in \mathbb{R}$, that is, y is not recurrent for Z.*

4. Some differences between classical and non classical dynamics

In this section we present some remarks and examples in the direction to understand the hypotheses demanded throughout the manuscript. First of all, we observe that, in order to assume preservation the measure, it is strictly necessary to impose the hypothesis

$$M \cap \varphi_t(\overline{\Sigma^e \cup \Sigma^s}) = \emptyset.$$

In order to see this, assume that μ is the Lebesgue measure and $M\cap\varphi_t(\overline{\Sigma^e\cup\Sigma^s})\neq\emptyset$. Suppose, without loss of generality, that $M\cap\varphi_t(\overline{\Sigma^e})\neq\emptyset$. In this case there exists a flow $f_t^{\Sigma^e}$, which remains on Σ^e, a measurable set E' and times $t_1,t_2\in\mathbb{R}$ such that $\mu(E')>\mu(f_t^{\Sigma^e}(E'))$, for all $t\in(t_1,t_2)$, since $f_t^{\Sigma^e}$ is contained in Σ^e and Σ^e has codimension one. As we commented before, this remark will be important in the statements of the main results in Section 5.

One can ask if the hypothesis mentioned above is too restricted, in the sense that it could lead to a classical behavior of the NSVF under study. However, that is not the case. Indeed, lots of applications in engineering leads to NSVF's having no sliding or escaping motions, that is, considering only sewing and tangency points (or even only sewing points), see [10] and references therein. Other known fact from NSVF's which is distinct to smooth dynamical system is that, surprisingly, even linear NSVF without sliding or escaping can present limit cycles (see, for instance, [16]). Another example of a system presenting no sliding or escaping but behaving very differently from smooth dynamical system is the following.

Example 4.1. *We consider the PSVF $Z=(X,Y)$ with $X(x,y)=(1,4x(1-x^2))$ and $Y(x,y)=(-1,4x(1-x^2))$. The phase portrait is exhibited in Figure 2. The PSVF Z has no sliding or escaping motion. However, the trajectory through the point presenting a simultaneous regular tangency is non-deterministic, which leads to a chaotic behavior (see more in [2]). Moreover, the simultaneous singular tangencies behave like a center. Despite of that, they do not preserve Lebesgue measure, which does not occur in smooth system.*

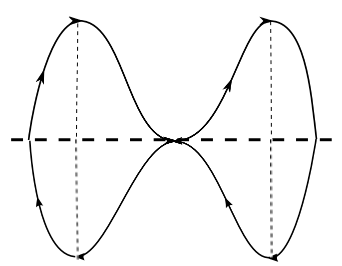

FIGURE 2. PSVF without sliding or escaping which present non-classical dynamics.

Previous example shows that the trajectory of every arc of orbit γ contained in the closed orbit behave as a nondeterministic way. Indeed, each point of γ which passes through the regular tangency when time goes to infinity should flows to one of the two possible arcs available for future trajectories. The conclusion is that, even considering PSVF's which avoid sliding or escaping, we can obtain non-determinism and uncountable trajectories.

In [2] the authors also presents a non-classical dynamical systems. Moreover, in that example they shows that

$$0<\mu\left(\varphi_t(\overline{\Sigma^e\cup\Sigma^s})\right)<\mu(M).$$

In other words, we can have systems whose measure of the saturation of $\overline{\Sigma^e\cup\Sigma^s}$ is positive, so the measure of $M\setminus\varphi_t(\overline{\Sigma^e\cup\Sigma^s})$ is not full. That is to say, classical

Poincaré theorems does not apply in such context, but the theorems presented in next section contemplate such situation.

Next example shows that Definition 3.3 is non-empty.

Example 4.2. *Consider μ the Lebesgue measure and $Z = (X, Y)$ the planar NSVF with switching manifold given by $\Sigma = F^{-1}(0)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ writes $F(x, y) = y$. Consider X and Y defined in Σ^+ and Σ^-, respectively, by $X(x, y) = (b, -a)$ and $Y(x, y) = (a, b)$, with $a < 0$ and $b > 0$. Observe that $(X.F(x, 0))\ (Y.F(x, 0)) = -ab > 0$, which means that every trajectory reaching Σ from Σ^- to Σ^+ crosses Σ in a sewing point. Moreover, the trajectories coming from Σ^- rotates ninety degrees in the clockwise direction when they cross Σ (see Figure 3). In this situation, it is easy to see that the global flow f_t associated to Z (which is unique since every point of Σ is a sewing one) is a map which preserves the measure μ of every measurable set of M, for all time $t \in \mathbb{R}$. Indeed, it holds since the rotation is a preserving-measure map (not only ninety degrees but any degree of rotation which preserves the sewing region). Therefore, according to Definition 3.3 the measure μ is invariant by Z. Observe that taking $b = -a$, we have a refracted system since $X.F(x, 0) = Y.F(x, 0) = -a$, which take an special place into non-smooth vector fields.*

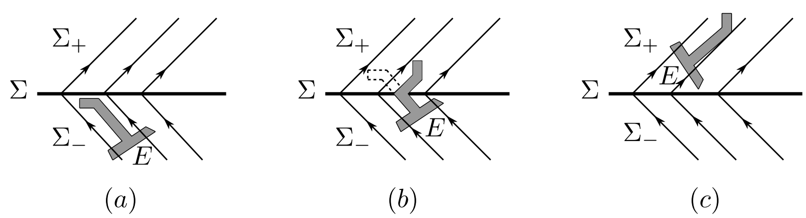

FIGURE 3. The global flow f_t of the NSVF preserves the measure of every measurable set E. From left to right: (a) the set E is entirely contained in Σ^- and has Lebesgue measure m; (b) the set E reaches Σ: the dashed line represents the virtual flow of E by the vector field Y which is reflected by the vector field X (filled drawing) preserving the Lebesgue measure of $E \cap \Sigma^+$; (c) the set E is totally reflected by vector field X and has the same original Lebesgue measure m.

5. Main Results

Before establishing our main results we need to prove some technical ones which shall help us in the proof of the theorems.

Lemma 5.1. *Consider $Z = (X, Y) \in \Omega$ and assume that Z has isolated tangencies and $M \cap \varphi_t(\overline{\sum^e \cup \sum^s}) = \emptyset$. Then the following holds:*

(a) *If X has a tangency point p of order even, then p is a tangency point for Y with order even, i.e. tangencies of order even appear pairwise.*
(b) *If X has a tangency point p of order odd, then there exists a neighborhood V_p of p in $\sum$ such that $Xf(q) \cdot Yf(q) \geq 0$, $\forall q \in V_p$.*

Proof. By hypothesis we have $Xf(p) = 0$ and $X^2f(p) \neq 0$. Assume $X^2f(p) > 0$, i.e. $Xf(q) < 0$ if $q < p$ and $Xf(q) > 0$ if $q > p$. In order to prove that p is a tangency point for Y we need to prove that $Yf(p) = 0$. By contradiction we suppose that $Yf(p) \neq 0$. Once $Yf(q) =< \nabla f(q), Y(q) >$ is a continuous function, there exists a neighborhood V_p of p in $\sum$ such that $Yf(q) > 0$ for all $q \in V_p$ or $Yf(q) < 0$ for all $q \in V_p$. Without loss of generality we can assume that $Yf(q) > 0$ for all $q \in V_p$. The other case is analogous. Then we get from the invariance of the signal of Yf that $Xf(q) \cdot Yf(q) < 0$ for all $q < p$, i.e. $\underline{V_p \cap \{x \in \sum; x < p\}} \subset \sum^s$, which is a contradiction to the fact that $M \cap \varphi_t(\overline{\sum^e \cup \sum^s}) = \emptyset$. Therefore $Yf(q) = 0$. This complete the proof of statement (a).

For statement (b) we can assume that $X^3f(p) > 0$, since p is a tangency point of order even for X. From the fact that tangencies are isolated and since $Xf(q)$ is a continuous function we get that there exists a neighborhood V_p of p of p in $\sum$ such that $Xf(q) < 0$ for all $q \in V_p \setminus \{p\}$. If $Xf(q) \cdot Yf(q) < 0$ for all $q \in V_p \setminus \{p\}$, then $V_p \cap \sum \subset \sum^s$ which is a contradiction. So $Xf(q) \cdot Yf(q) \geq 0$. □

A consequence of Lemma 5.1 is the result which follows and its proof is straightforward from the fact that there are open sets $(t_i, t_{i+1}) \subset \sum^s$, where $t_k \in \sum^t$, for all k, since tangencies are isolated.

Corollary 5.1. *Under the same hypotheses of Lemma 5.1 if p is a tangency point of order odd for X then every orbit passing through a point in V_p is unique.*

It is important to note that, in Corollary 5.1, since the order of the tangencies is odd, the trajectory through them are unique.

Taking into account the previous definitions and considerations we are able to state and prove the Poincaré Recurrence Theorem in two versions, one measurable and another topological. First we introduce the measurable version of such theorem, as follows:

Theorem 5.1. *Consider $Z \in \Omega$ and assume that Z has a finite number of tangencies, $M \cap \varphi_t(\overline{\sum^e \cup \sum^s}) = \emptyset$ and Z has no equilibrium points. Let μ be a finite measure in M which is invariant by some measurable flow $f_t : M \to M$ of Z. If $E \subset M$ is a measurable subset of M with $\mu(E) > 0$ then for $\mu-$almost every point $x \in E$ there exist sequences $(t_i)_{i\geq 1}$ and $(s_i)_{i\geq 1}$ satisfying:*

(1) $t_1 < s_1 < t_2 < s_2 < \ldots < t_i < s_i < \ldots$, *for all* $i \geq 1$
(2) $f_t(x) \in E$, $\forall t \in [t_i, s_i]$, $\forall i \geq 1$
(3) $t_i \to \infty$ *when* $i \to \infty$
(4) $s_i \to \infty$ *when* $i \to \infty$

Before provide the proof of Theorem 5.1 we stress that it is possible to allow equilibrium points in M if they are particular centers, in the sense M posses only preserving measures' centers. That happen, for instance, if $Z = (X, Y)$ with $X = Y$ has a linear center. Otherwise, centers in PSVF's does not preserve measure, which is an obvious difference between smooth dynamical systems and non-smooth ones.

Proof. Let $E \subset M$ be a measurable subset of M with $\mu(E) > 0$. Consider the set

$$E_0 = \{x \in E : \ f_t(x) \notin E \ \ \forall\, t \geq 1\}.$$

Hence, if a point $x \in E$ remains a finite time inside E, i.e. there exists $T_x > 0$ such that $f_t(x) \notin E$ for all $|t| \geq T_x$, then $x \in E_0$ or $x \in f_1^{-k}(E_0)$ for some positive integer k.

In order to prove the theorem we just need to prove that

$$\mu(\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)) = 0$$

and for this we need to assure that all the preimages of E_0 by f_1 are pairwise disjoint, i.e. $f_1^{-i}(E_0) \cap f_1^{-j}(E_0) = \emptyset$, for all $i \neq j$. Firstly we suppose that $\sum \cap \left[\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)\right] = \emptyset$.

Since, for each fixed t, the map $f_t(\cdot)$ is a diffeomorphism it is not difficult to see that the preimages of E_0 by f_1 are pairwise disjoint, i.e. $f_1^{-i}(E_0) \cap f_1^{-j}(E_0) = \emptyset$, for all $i \neq j$.

Consequently, since μ is invariant by f_t for all t, in particular for f_1, we get

$$\mu(\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)) = \sum_{k=0}^{\infty} \mu(f_1^{-k}(E_0)) = \sum_{i=k}^{\infty} \mu(E_0).$$

Furthermore, once μ is finite we have $\sum_{i=0}^{\infty} \mu(E_0) < \infty$ which implies $\mu(E_0) = 0$ and then $\mu(\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)) = 0$.

Now we shall deal with the case $\sum \cap \left[\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)\right] \neq \emptyset$. Firstly we observe that, in this case, the hypothesis $M \cap \varphi_t(\overline{\sum^e \cup \sum^s}) = \emptyset$ implies that in $\sum$ we can have only tangencies or sewing region. Indeed, when the intersection $\sum \cap \left[\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)\right]$ occurs in a sewing region we apply the previous argument. This is possible because the trajectory passing through a point in the sewing region is unique even if it is non-smooth.

Now we assume that the intersection $\sum \cap \left[\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)\right]$ occurs in a tangency point $p \in \sum$. In this case we have to analyze two situations, as described in Lemma 5.1. That is, we have to consider the order of the tangency. Nevertheless, if p is a tangency point of order even, using Lemma 5.1(a), since tangencies appear pairwise, it lead us to the following possibilities for p as a tangency point for X and Y:

(i) p is a singular tangency for both X and Y;
(ii) p is a regular tangency for X and a singular one for Y (or vice-versa);
(iii) p is a regular tangency for both X and Y.

Case (i) can not happen since a common singular tangency generates a point which behaves like a smooth center or a smooth focus in the sense that the trajectory of such a point is stationary and attracts or repels every point in a neighborhood (case focus) provides a continuum of periodic orbits not necessarily preserving measure. In any case we have a contradiction to the hypothesis which does not allow equilibrium points.

Case (ii) is similar to the case where we have only regular or sewing points once the trajectory through such points is unique, that is, the local trajectory passing by p remains on the same region before and after reach p. So previous argument apply since the trajectory through p is just an arc of orbit of X or Y.

Case (iii) is not so trivial once there is no uniqueness of trajectory at points which are regular tangencies for both X and Y. That is true because p is reached in finite time for these vector fields, so the set $\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)$ has a non-deterministic nature. However, it can be fixed by considering the set

$$\widetilde{E} = E \setminus \bigcup_{p \in \Sigma^t} \varphi_t(\{p\}).$$

Indeed, since there are two possibilities for the trajectory passing at $p \in \sum^t$ and, by hypothesis, there exists a finite number of tangencies, the measure of the saturation of p is zero, so it is the union of saturation of the finite quantity of regular tangencies. Therefore, we get

$$\mu(E) = \mu(\widetilde{E}),$$

that is, we can avoid these kind of tangencies and apply the previous results for the set $E \setminus \widetilde{(E)}$.

Now, if p is a tangency point of order odd, using Corollary 5.1, once the trajectory is unique in every odd tangency point, we can use similar arguments as in the sewing case above. Other cases taking into account odd order tangencies are not possible by the hypothesis $M \cap \varphi_t(\overline{\sum^e \cup \sum^s}) = \emptyset$.

Finally, if $\Sigma \cap \left[\bigcup_{k=0}^{\infty} f_1^{-k}(E_0)\right] \subset \left(\Sigma^S \cup \Sigma^t\right)$, then we can combine the previous arguments to conclude that the proof goes analogously.

□

The next result establishes a topological version of the Poincaré Recurrence Theorem. This version is similar to the classical scenario of smooth vector fields but we shall repeat it here for completeness of the text and since it is very short and simple to understand.

Theorem 5.2. *Let $Z = (X, Y)$ be a non-smooth vector field defined on a compact manifold M. Let μ be a finite measure in M which is invariant by Z. Then, μ-almost every point $x \in M$ is recurrent by Z.*

Proof. Consider $(U_k)_{k \geq 1}$ a basis of open sets U_k, for all $k \geq 1$, of M. Since μ is invariant by Z, by definition (3.3), μ is invariant by every flow $f_t : M \to M$. Then for each $k \geq 1$ we consider the set

$$\tilde{U}_k = \{x \in U_k : \exists\, T_k > 0 \text{ such that } \{f_t(x); |t| \geq T_k\} \cap U_k = \emptyset\}.$$

Applying Theorem 5.1 for each set $\tilde{U}_k$ we conclude that $\mu(\tilde{U}_k) = 0$, for every $k \geq 1$, and then $\mu(\bigcup_{k=1}^{\infty} \tilde{U}_k) = 0$.

In order to prove the theorem we just need to prove that every point $x \in M \setminus \bigcup_{k=1}^{\infty} \tilde{U}_k$ is recurrent by Z. For this, we take a point $x \in M \setminus \bigcup_{k=1}^{\infty} \tilde{U}_k$. Then $x \notin \tilde{U}_k$ for all $k \geq 1$. We note that $x \notin \tilde{U}_k$ means that for every $T_k > 0$ there exists $t_k > T_k$ such that $f_{t_k}(x) \in U_k$. But given a neighborhood U of x there exists an open set U_k from the basis $(U_k)_{k \geq 1}$ such that $x \in U_k \subset U$. Thus $x \in U_k \setminus \tilde{U}_k$, that is, x is recurrent by Z.

□

6. Conclusions

Here we presented some concepts related to ergodic features of non-smooth vector fields. We introduced the ideas of invariant measures and recurrence in this context and presented an important result coming from the ergodic theory of the dynamical systems, namely, the Poincaré Recurrence Theorem. We understand that establishing such a result in the NSVF's scenario improve the theory of dynamical systems in two ways. First, it should be of interest in ergodic theory once the results presented throughout the current paper allow us to soften the hypotheses of a classical theorem in the sense that it provides the result by considering multi-valued flows which, besides, are usually non-regular. As far as the authors know, this is the first result improving the Poincaré Recurrence Theorem by allowing such degeneracies, which means, among other considerations, that this theorem can be applied for a class of maps much larger than considered at the present moment. Second, the results presented in this work could provide some advance in the theory of non-smooth vector fields since they establish a classical result of the theory of dynamical systems in the context of a very recent theory, that is, considering non-smooth vector fields, theory which has found a wide range of applications in many real problems. Nevertheless, translate adapted versions of classical theorems or testing their validity into the scenario of NSVF is one of the main goals of this theory. More specifically, in what concerns ergodic aspects of NSVF, the present paper is one of the first works dealing with this aspect and it comes after some few works which can be enumerated by [2], [3] and [7], and it goes forward in parallel to the works [8] and [15]. Future works, some of then in progress by some of the authors cited throughout this paper, should address other aspects of ergodic theory as, for instance, the acclaimed $\lambda-$lemma, apart from introducing the basic statements and definitions of ergodic theory for NSVF.

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