Global asymptotic analysis of planar Filippov vector fields

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November 4, 2019

Abstract
In this paper the structure of global trajectories of a class of planar piecewise-smooth vector field is studied. We assume that it is defined by two $C^r$ vector fields whose trajectories are separated by a smooth curve $\Sigma$. For our purposes we also assume confinement of trajectories on a compact set $K$. In this scenario the behavior of the global trajectories is fully analyzed and their limit sets are classified. Moreover, the trajectories are allowed to switch from $\Sigma$ to outside it infinitely many times through sliding or escape regions. Consequently we are able to present a version of the Poincaré-Bendixson Theorem for piecewise-smooth vector fields without avoiding sliding. Some (limit) sets presenting sliding motion or even non-empty interior are also obtained. In the last case, we are able to prove the existence of nondeterministic chaos. Besides limit sets, from the global analysis we distinguish important features of the trajectories in terms of the tangency points. Some applications for the class of piecewise-linear vector fields fitting our hypotheses are also presented.

1 Introduction
The dynamic of a PSVF is generally more complicated than a smooth vector field because there are several ingredients playing some role. In fact, to fix ideas, even in the simplest case having two vector field whose trajectories are separated by a common frontier, one must consider not only the dynamic of each particular vector field but also their contact to the boundary. Moreover, the trajectories of such vector fields eventually slide on the boundary by an amalgamation processes occurring at the moment of the collision generating new trajectories. For that reason the first steps toward the construction of a consistent theory of PSVF require the establishing and validation of new results analogous to classical ones. In this direction we highlight some of those results already established into the non-smooth context in the areas of Stability [2], Chaos [4, 5], Bifurcation [11, 12], Closing Lemma [6], Limit Sets [3] and the Peixoto’s Theorem [10].

A particular gap to a whole comprehension of Filippov vector fields is the lack of global results. Effectively in many cases local aspects must be assumed. This is particular important when trajectories interact to the boundary of the manifold since results as submersion theorem simplify coordinates and consequently calculations. In our context, however, even performing an miscellaneous approach from analysis and geometry, no local aspects are required but only contact conditions on a compact portion of the phase portrait, say $K$. More specifically the class of PSVFs we consider is defined on two regions separated by a regular co-dimension one manifold $\Sigma$, being the critical elements on $K$ finite and therefore isolated. Finally, we assume that $K$ is invariant for the Filippov flow in the sense of the Poincaré-Bendixson Theorem.

The goal of this paper is to study global asymptotic aspects of a maximal trajectory $\Gamma_Z(t,p)$ of a Filippov system (see Subsection 3.1). More precisely we are interested in obtaining the limit sets of such trajectories. The main result of the paper provide a fine classification of those objects by using the classical Poincaré-Bendixson Theorem and a generalization of the results obtained in [3]. The results we present are, however, more general due to the fact that it is allowed the so called sliding motion. That assumption of course turns the problem more complicate but on the other hand the conclusions are not only richer but also more interesting. Additionally, we verify that the limit set may present chaotic behavior.
This paper is organized as follows. In Section 2 we present the main results and a brief discussion of them. In Section 3 we present the fundamental notions of PSVF (Subsection 3.1) and we discuss some extensions of Filippov vector fields for transitions between sliding and escaping regions (Subsection 3.2). In Section 4 we analyze the behavior of a maximal trajectory contained on a compact set $K$. We also establish auxiliaries results concerning pseudo-cycles (Subsection 4.1), mild pseudo-cycles and chaotic sets (Subsections 4.2 and 4.3). In Sections 5 and 6 we prove the main results of the paper and provide some applications for piecewise-linear vector fields, respectively. Finally in Section 7 we present a conclusion on the main achievements of the paper.

2 Main results and Discussions

In this paper we develop a global study of PSVF through an asymptotic analysis of their trajectories in order to study limit sets and their properties. The results we obtain may be outlined as follows:

- The authors in [3] provide a particular version of Poincaré-Bendixson theorem to PSVF by allowing discontinuities but avoiding visits to the sliding/escaping region. Under such weaken hypotheses, besides the well known limit sets from classical theory the authors obtained crossing and tangential-crossing pseudo-cycles, pseudo-graphs and tangency points of type I (cf. Definitions on Section 3.1, 4.1 and 4.3).

- In the present paper we first assume that trajectories transit from and to the switching manifold $\Sigma$ until they remain outside $\Sigma$ or inside it. In the first situation, we show that no additional limit set emerges to the list provided in [3]. In the second situation, however, we add to that list the pseudo-equilibria of Filippov vector fields and tangency points of type II, see Fundamental Lemma.

- Posteriorly we assume infinitely many transitions on $\Sigma$. In this case we increase the list given in Fundamental Lemma by adding sliding and tangential-sliding pseudo-cycles as well as chaotic sets, see Theorem 1.

- Finally, we explore some interesting properties involving nontrivial recurrences of the chaotic sets we obtained as limit sets, see Theorem 2.

We now establish the main results of the paper. The objects we are going to describe within the results are detailed on Section 3.

Theorem 1. Let $Z = (X,Y)$ be a planar PSVF and $\Sigma$ a smooth manifold separating $\mathbb{R}^2$ into two unbounded regions. Assume that $Z$ has a maximal trajectory $\Gamma_Z(t,p)$ whose positive trajectory $\Gamma^+_Z(t,p)$ is contained on a compact subset $K$. Assume also that $X$ and $Y$ have a finite number of critical points on $K$, each $X$ and $Y$ has at most one tangency point of finite order on $K \cap \Sigma$ and $Z$ has only isolated pseudo-equilibria on $K \cap \Sigma$. Then the $\omega$-limit set $\omega(\Gamma_Z(t,p))$ of $\Gamma_Z(t,p)$ is one of the following objects:

(i) an equilibrium of $X$ or $Y$;

(ii) a periodic orbit of $X$ or $Y$;

(iii) a graph of $X$ or $Y$;

(iv) a pseudo-equilibrium of $Z$;

(v) a (crossing, tangent or sliding) pseudo-cycle of $Z$;

(vi) a mild pseudo-cycle of type I, II or III of $Z$;

(vii) a pseudo-graph of $Z$;

(viii) a tangency of type I or II;

(ix) a chaotic set of type III.

We highlight some points concerning the previous theorem. First, the sliding pseudo-cycles may have distinct topological types. For instance, some of the sliding pseudo-cycles we obtain are contained on $\Sigma^+$ (or $\Sigma^-$) and possesses only a segment of slide. However, we also obtain sliding limit cycles occupying both $\Sigma^+$ and $\Sigma^-$ as well as having two disjoint sliding regions. These particular cases, some bifurcations phenomena are observed (see Section 7). We also notice in statement (ix) that it may occur a $\omega$-limit set having nonempty interior and other interesting topological properties, as presented in the next theorem.
Theorem 2. Let $Z$ be as in Theorem 1 and assume that $Z$ has a parabolic or hyperbolic tangency point $p_0 \in \partial(\Sigma^c \cup \Sigma^{s})$. Assume also that there exist a trajectory passing by both $\Sigma^s$ and $\Sigma^c$ infinitely many times. Then $Z$ admits a chaotic set $\Lambda$ of type III as $\omega$-limit of a maximal trajectory $\Gamma_{Z}(t,p)$ in such way that every point on its interior

(a) is periodic;
(b) admits a dense orbit on $\Lambda$ passing through it;
(c) presents non-trivial recurrence;
(d) is a non-trivial non-wandering point.

In order to prove Theorems 1 and 2 we have to state the following generalization of Theorem 1 of [3].

Fundamental Lemma. Let $Z = (X,Y)$ be a planar PSVF and assume that $Z$ has a maximal trajectory $\Gamma_{Z}(t,p)$ whose positive trajectory $\Gamma_{Z}^{+}(t,p)$ is contained on a compact set $K$. Assume also that $X$ and $Y$ have a finite number of critical points on $K$. If there exist $t_0 \in \mathbb{R}$ such that $\Gamma_{Z}(t,p) \not\in \Sigma^c \cup \Sigma^{s}$, $\forall t > t_0$ then the $\omega$-limit set $\omega(\Gamma_{Z}(t,p))$ of $\Gamma_{Z}(t,p)$ is one of the following objects:

(i) an equilibrium of $X$ or $Y$;
(ii) a periodic orbit of $X$ or $Y$;
(iii) a graph of $X$ or $Y$;
(iv) a crossing pseudo-cycle of $Z$;
(v) a mild pseudo-cycle of type II of $Z$;
(vi) a pseudo-graph of $Z$;
(vii) a tangency of type II.

On the other hand, if $\Gamma_{Z}(t,p) \in \Sigma^c \cup \Sigma^{s}$ for $t$ sufficiently large then $\omega(\Gamma_{Z}(t,p))$ is one of the following objects:

(viii) a pseudo-equilibrium of $Z$ or
(ix) a tangency of type II.

Remark 1. In order to prove Theorems 1 and 2 and the further results of the paper we will consider a maximal trajectory $\Gamma_{Z}(t,p)$ contained on $K$ in such way that there exists $s_n \to +\infty$ and $t_n \to +\infty$ satisfying $\Gamma_{Z}(s_n,p) \in \Sigma^c \cup \Sigma^{s}$ and $\Gamma_{Z}(t_n,p) \not\in \Sigma$. In other words, we will assume that $\Gamma_{Z}(t,p)$ visits and leaves $\Sigma^c \cup \Sigma^{s}$ infinitely many times. Notice that in the second situation we may apply Fundamental Lemma straightforwardly.

It worth to mention that the shape of $\Sigma$ does not play any role for the results as long as the hypotheses are verified. We also assume the natural ordering for $\Sigma = (-\infty, +\infty)$. Moreover, for our purposes we only consider tangency points of even order. Otherwise the trajectories crosses $\Sigma$ always in the same direction as the regular case being the last situation considered along the paper.

From now on we assume that trajectories of the Filippov vector field composing a maximal trajectory $\Gamma_{Z}(t,p)$ are contained on the compact set $K$ for positive values of $t$.

3 Preliminaries

3.1 Piecewise-smooth vector fields

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function having $0 \in \mathbb{R}$ as regular value. We call $\Sigma = f^{-1}(0)$ the switching manifold that splits the plane into the two unbounded regions $\Sigma^- = \{(x,y) \in \mathbb{R}^2 ; f(x,y) \leq 0 \}$ and $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 ; f(x,y) \geq 0 \}$. We consider the piecewise-smooth vector fields $Z = (X,Y)$ on $\mathbb{R}^2$ defined by

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } f(x,y) \leq 0, \\ Y(x,y), & \text{if } f(x,y) \geq 0, \end{cases}$$

where $X$ and $Y$ are smooth vector fields defined on $\mathbb{R}^2$. Now, in order to study PSVF on $\Sigma$ we must introduce some notations. Indeed, given a vector field $X$ on $\mathbb{R}^2$, we consider the Lie derivatives $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and $X^1.f(p) = \langle \nabla X^i.f(p), X(p) \rangle$, $i \geq 2$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$. On $\Sigma$ we distinguish three regions satisfying $(X.f(p)) \cdot (Y.f(p)) \neq 0$. i) The sewing region $\Sigma^c$ formed by the points $p \in \Sigma$ such that the trajectory of $X$ (resp. $Y$) meets $p$ in finite future time and the trajectory of $Y$ (resp. $X$) meets $p$ in finite past time. ii) The escaping region $\Sigma^{s}$ formed by the points $p \in \Sigma$ such that the trajectories of $X$ and $Y$ meet $p$ in finite past time. Lastly iii) the sliding region $\Sigma^{s}$ formed by the points $p \in \Sigma$ such that the trajectories of $X$ and $Y$ meet $p$ in finite future time.
The points \( p \in \Sigma \) such that \( X.f(p) = 0 \) (resp. \( Y.f(p) = 0 \)) are called tangency points of \( X \) (resp. \( Y \)). We say that a tangency point \( p \) has contact of order \( n \in \mathbb{N} \) if the Lie derivatives \( X^k.f(p) \) vanish for \( k < n \) and \( X^n.f(p) \neq 0 \). We also classify tangency points according to the following: let \( X \) and \( Y \) two smooth vector fields. We say that \( p \in \Sigma \) is an invisible tangency if the contact order \( r \) of \( \Gamma_X \) (resp. \( \Gamma_Y \)) passing through \( p \) is even and \( X^r.f(p) < 0 \) (resp. \( Y^r.f(p) > 0 \)). On the other hand, we say that \( p \in \Sigma \) is a visible tangency if the contact order \( r \) of \( \Gamma_X \) (resp. \( \Gamma_Y \)) passing through \( p \) is even and \( X^r.f(p) > 0 \) (resp. \( Y^r.f(p) < 0 \)).

For our purposes it is also interesting to consider a special configuration of tangency point, namely the case where the vector fields share a tangency. We call that points double tangency points. For a PSVF \( Z = (X,Y) \) we say that a double tangency is elliptical if it is an invisible tangency for both \( X \) and \( Y \), parabolic if it is a visible tangency of even order for a vector field and invisible one for the other and hyperbolic if the double tangency is visible of even order for \( X \) and \( Y \) (see Figure 1). In this paper we also refer to elliptical tangency points by tangency of type I. On the other hand, let \( p \in \Sigma \) an invisible tangency for \( X \). If (i) \( p \) is on the boundary of a sliding region and attracts sliding orbits and (ii) \( p \) is a tangency of odd order for \( Y \) then we call \( p \) a tangency of type II (see Figure 2). We note that if \( p \) satisfies the previous conditions except (ii) and it is regular for \( Y \) then \( p \) repels sliding orbits.

In order to deal with sliding motion in the PSVFs we adopted the Filippov convention according to [8] to define a vector field \( Z^\Sigma \) on \( \Sigma^e \cup \Sigma^s \):

\[
Z^\Sigma(p) = \frac{X.f(p).Y(p) - Y.f(p).X(p)}{X.f(p) - Y.f(p)}, \quad p \in \Sigma^e \cup \Sigma^s.
\]

In this paper we call \( Z^\Sigma \) the Filippov vector field. The points \( p \in \Sigma^e \cup \Sigma^s \) such that \( Z^\Sigma(p) = 0 \) are called pseudo equilibrium of \( Z \) and the trajectories of \( Z^\Sigma \) may constitute a trajectory of \( Z \). We alert that other convention could lead to different results, see for instance [7, 2].

**Definition 1.** A global trajectory \( \Gamma_Z(t,p) \) of a piecewise-smooth vector field \( Z \) is the trace of a continuous curve obtained by concatenation of trajectories of \( X \) and/or \( Y \) and/or \( Z^\Sigma \). A maximal trajectory \( \Gamma_Z(t,p) \) is a global trajectory that cannot be extended by any concatenation of trajectories of \( X, Y \) or \( Z^\Sigma \).

**Definition 2.** Given \( \Gamma_Z(t,p) \) a maximal trajectory of \( Z \), the set

\[
\omega(\Gamma_Z(t,p)) = \{ q \in \mathbb{R}^2 : \exists (t_n) \subset \mathbb{R} \text{ satisfying } \lim_{t_n \to \infty} \Gamma_Z(t_n,p) = q \}
\]

is called \( \omega \)-limit set of \( \Gamma_Z(t,p) \).

A more detailed presentation of trajectories of PSVFs can be found in [11].

### 3.2 Extension of Filippov vector fields beyond the boundary of \( \Sigma^e \)

In what follows we briefly discuss the extension of one-dimensional vector fields to the boundaries of their domains in order to allow infinitely many transitions over \( \Sigma \).

Indeed, consider \( Z \) as in equation (1) and \( Z^\Sigma \) the associated Filippov vector field. We shall extend...
the Filippov vector field to the closure of escaping and sliding regions, that is, to their adjacent tangency points if it is possible. Assume that \( p \) is the common boundary between \( \Sigma^e \) and \( \Sigma^c \) and consider
\[
L_{c,s} = \lim_{q \to p} Z(q).
\]
The limit is consider at \( q \in \Sigma^c \) for \( L_c \) and \( q \in \Sigma^s \) for \( L_s \). We split the analysis into two cases:

(a) \( L_c = L_s \). In this case we define
\[
L = \lim_{q \to p} Z(q) = L_c = L_s,
\]
and therefore \( Z(p) = L \), so it remains to analyze if \( p \) is a pseudo-equilibrium or not. Indeed, let \( U_p \) be a neighborhood of \( p \) and \( U = (U_p \cap \Sigma) \setminus \{ p \} \). If \( L \neq 0 \) then the Filippov vector field in \( U \) points to the same direction than \( L_{c,s} \) and therefore the orbit flows through \( p \) from \( \Sigma^e \) to \( \Sigma^s \) or vice-versa. In other words, \( p \) is a regular point for the extended Filippov vector field. If \( L = 0 \) then \( p \) is a pseudo-equilibrium for it.

(b) If \( L_c \neq L_s \) and both are non-zero pointing to the same direction, then by a re-scaling of time of \( Z^e \) or \( Z^c \) we obtain the same situation of (a) and we are done. The most interesting situation occurs when the vector fields \( Z^e \) and \( Z^c \) point to opposite directions. In this case we cannot extend the Filippov vector field to \( p \) because we have a one-dimensional Filippov dynamics. The same happens when \( L_c = 0 \) or \( L_s = 0 \), i.e., the Filippov vector field cannot be extended beyond its boundary.

Remark 2. Notice that using the expression of the Filippov vector field given in (2) if \( p \in \Sigma \) is an equilibrium point of \( Y \) or \( X \) then \( \lim_{q \to p} Z(q) = 0 \) and therefore \( p \) is an equilibrium point of the extended Filippov vector field.

4 Global analysis and auxiliaries results

This section is devoted to analyze the possible limit sets arising in the context of Theorem 1. In order to highlight the different kind of objects we are going to deal with, we split the study into some particular cases starting with the pseudo-cycles.

4.1 Pseudo-cycles

In this subsection we fully describe how a pseudo-cycle (of crossing, tangent or sliding type) emerges as \( \omega \)-limit set of a maximal trajectory as stated in Theorem 1. In particular, as commented before, the approach we use allow us to better comprehend the topological structure of pseudo cycles as well as their asymptotic behavior.

Definition 3. Consider \( Z \) a piecewise-smooth vector field. Consider \( \Gamma \) a closed maximal trajectory of \( Z \) such that \( \Gamma \cap \Sigma \neq \emptyset \) and assume that it does not contain neither equilibria nor pseudo-equilibria. Then, we say that \( \Gamma \) is a pseudo-cycle (see Figure 3) if

(i) \( \Gamma \) is positively or negatively invariant and

(ii) \( \Gamma \) does not contain a proper maximal trajectory.

The last definition is a refinement of the concept of pseudo cycle usually considered in the literature. That allow us to explore the pseudo cycles in a more accurate way. In particular we distinguish three types of pseudo-cycles: (a) crossing pseudo-cycles, which are those satisfying \( \Gamma \cap \Sigma \subset \Sigma^c \), see Figure 3 (a), (b) tangent pseudo-cycles, when \( \Gamma \cap \Sigma^t \neq \emptyset \) and \( \Gamma \) does not present sliding motion, see Figure 3 (b1 − 2), and (c) sliding pseudo-cycles, which are the pseudo-cycles having sliding motion, see Figure 3 (c1 − 3).

Since the crossing pseudo-cycles were addressed in Fundamental Lemma, next we will restrict our attention to the tangent and sliding pseudo-cycles. We do that analyzing the pseudo cycles in terms of the configuration of the tangency points. Indeed, let \( \Gamma_Z(t,p) \) be a maximal trajectory of \( Z \) contained on a compact set \( K \). We will also assume that \( \Gamma_Z(t,p) \) leaves and returns to \( \Sigma^t \cup \Sigma^c \) infinitely many times as considered previously.

4.1.1 The regular-tangent case

Assertion 1. Assume that hypotheses of Theorem 1 hold, \( X \) has a tangency point \( p^- \) but \( Y \) is transversal at \( \Sigma \) on \( K \). Then \( \Gamma_Z(t,p) \) can be taken such that \( \omega(\Gamma_Z(t,p)) \) is a tangent or a sliding pseudo-cycle.
Proof. Without loss of generality we take $p^- = (0,0)$. As commented before we can avoid odd tangency points so the following possibilities may occur.

(i) $\Sigma^e = (-\infty,0)$ and $\Sigma^c = (0,+\infty)$;
(ii) $\Sigma^e = (-\infty,0)$ and $\Sigma^c = (0,+\infty)$;
(iii) $\Sigma^e = (-\infty,0)$ and $\Sigma^c = (0,+\infty)$;
(iv) $\Sigma^e = (-\infty,0)$ and $\Sigma^c = (0,+\infty)$.

Consider $p \in K$. First we assume that statement (i) holds. Since $\Gamma^X(t,p)$ leaves and returns to $\Sigma^c$ infinitely many times, we get $\Gamma^X(t,p) \cap \Sigma \subset \Sigma^c \cup \{0\}$ being 0 a visible tangency for $X$ and a regular point for $Y$. Moreover, clearly the trajectory $\Gamma^X(t,p)$ only returns to $\Sigma$ by 0, so let $q^-_1 \geq 0$ be the point on $\Sigma$ such that the negative trajectory of $X$ starting at 0 meets $\Sigma$ (see Figure 3 (c1)). Since we are assuming infinitely many returns to $\Sigma$, it follows that $\Gamma^X(t,p)$ only leaves $\Sigma^e$ by $q^-_1$. Consequently we obtain a tangent pseudo-cycle of $Z$ if $q^-_1 = 0$ (whose intersection with $\Sigma$ is the tangency point 0) or a sliding pseudo-cycle of $Z$ if $q^-_1 > 0$ and we are done.

It is easy to see that statement (ii) leads to the analogous situation. Similarly, the proof of statements (iii) and (iv) is treated in the same way by reversing time in the trajectory $\Gamma_Z(t,p)$. 

4.1.2 The tangent-tangent case assuming $p^- \neq p^+$

Now each vector field $X$ and $Y$ contribute with tangency points $p^-$ and $p^+$, respectively. Without loss of generality, assume that $p^- < p^+$. Now we have the following configurations of $\Sigma$:

(i) $\Sigma^e = (-\infty,p^-) \cup (p^+,+\infty)$ and $\Sigma^c = (p^-,p^+)$;
(ii) $\Sigma^e = (-\infty,p^-) \cup (p^+,+\infty)$ and $\Sigma^c = (p^-,p^+);
(iii) $\Sigma^e = (-\infty,p^-)$, $\Sigma^c = (p^-,p^+)$ and $\Sigma^c = (p^+,+\infty);
(iv) $\Sigma^e = (-\infty,p^-)$, $\Sigma^c = (p^-,p^+)$ and $\Sigma^c = (p^+,+\infty)$.

We also split the analysis according to the visibility or not of $p^-$ and $p^+$. Since $\Gamma^X_Z(t,p)$ switches from and to $\Sigma$ (see Remark 1), the invisible-invisible case cannot occur. In fact, if $p^-$ and $p^+$ are invisible tangency points then either $\Gamma^X_Z(t,p) \in \Sigma^c$ or $\Gamma^X_Z(t,p) \not\in \Sigma$ for $t$ sufficiently large. Then we only analyze the cases where at least one tangency point $p^-$ or $p^+$ is visible.

The invisible-visible sub-case

Assertion 2. Assume that hypotheses of Theorem 1 hold and both $X$ and $Y$ have non coincident tangency points $p^-$ and $p^+$ on $K \cap \Sigma$, respectively, having opposite visibility. In this case $\Gamma^X_Z(t,p)$ can be taken such that $\omega(\Gamma^X_Z(t,p))$ is a crossing, a tangent or a sliding pseudo-cycle.

Proof. Without loss of generality we suppose that $p^-$ is an invisible tangency point and $p^+$ is a visible one. Assume that statement (i) holds. Also $\Gamma^X_Z(t,p)$ leaves and returns to $\Sigma$, infinitely many times. So there exists a value $\ell > 0$ satisfying $\Gamma^X_Z(t,p) = t^+$. Also, there exists a point $p^+_1 \leq p^+$ which is the first return on $\Sigma$ such that the positive trajectory of $Y$ from $p^+$ meets $\Sigma$. (See Figure 4).

We have three situations:

- If $p^-_1 \in (p^-,p^+]$ then the regular-tangent case in the previous subsection applies. So we obtain a tangent or sliding pseudo-cycle of $Z$ if $p^-_1 = p^+$ or $p^-_1 < p^-_1 < p^+$, respectively. (See Figure 4 (a)).
Remark 3. There is a particular difference between the tangent and sliding pseudo-cycles appearing in Subsection 4.1.1 and the previous one. The tangent pseudo-cycle that appear in Subsection 4.1.1 touches \( \Sigma \) in a single point while those appearing in previously may touch \( \Sigma \) in two points (one of tangency type and other of crossing type). Moreover the sliding pseudo-cycles of Subsection 4.1.1 are entirely contained on \( \Sigma^+ \) or \( \Sigma^- \) while those appearing in this subsection may occupy both regions \( \Sigma^+ \) and \( \Sigma^- \).

The visible-visible sub-case

Assertion 3. Assume that hypotheses of Theorem 1 hold, \( X \) and \( Y \) have non coincident tangency points \( p^- \) and \( p^+ \) on \( K \cap \Sigma \), respectively, being both visible. Then \( \Gamma_Z(t,p) \) can be taken such that \( \omega(\Gamma_Z(t,p)) \) is a crossing, a tangent or a sliding pseudo-cycle.

Proof. Let \( p \) be a point on \( K \). Assume that statement (i) holds and \( \Gamma_Z(t,p) \) visits and leaves \( \Sigma^- \) infinitely many times. Since both \( p^+ \) and \( p^- \) attract trajectories of the Filippov vector field there exist at least one pseudo equilibrium point between \( p^- \) and \( p^+ \). Moreover the trajectory of a point \( q \in \Sigma^s \) only leaves \( X \) by \( p^- \) or by \( Y \) at \( p^+ \). Suppose that \( \Gamma_Z(t,p) \) leaves \( \Sigma^s \) at \( p^- = \Gamma_Z(t,p) \) for some \( t > 0 \) and let \( p^- \geq p^- \) on \( \Sigma \) be the first point where the positive trajectory of \( X \) from \( p^- \) meets \( \Sigma \).

Now we have three situations to consider in terms of the tangency points. We start assuming \( p^- \geq p^+ \) (see Figure 5). Let \( p^*_2 \in \Sigma \) be the first point such that the positive trajectory of \( Y \) from \( p^- \) meets \( \Sigma \). Then \( p^- \leq p^- \leq p^+ \) since \( \Gamma_Z(t,p) \) returns to \( \Sigma^- \) infinitely many times. In fact if \( p^*_2 \leq p^- \) then \( \Gamma_Z(t,p) \) only touches \( \Sigma \) in sewing points for sufficiently large values of \( t \) and therefore the Fundamental Lemma applies. Moreover, clearly the situation \( p^- > p^+ \) cannot occur. If \( p^- = p^- \) then we obtain a tangent pseudo-cycle whose intersection with \( \Sigma \) is the tangency point \( p^- \) and the tangency point \( p^- = p^+ \) or a sewing point \( p^- > p^+ \). If \( p^- = p^+ \) then we obtain a tangent pseudo-cycle whose intersection with \( \Sigma \) is the tangency point \( p^+ \). Now suppose that \( p^- < p^- < p^+ \). Since \( \Gamma_Z(t,p) \) visits \( \Sigma^- \) infinitely many times then \( p^- \) cannot be located between two pseudo-equilibrium points. Consequently the Filippov vector field connects \( p_- \) to either \( p^- \) or \( p^+ \). The first case is exhibited in the Figure 5 (a). In the second case the positive trajectory of \( Y \) connects \( p^+ \) to \( p^*_2 \) (see Figure 5 (b)). In the first case we obtain a sliding pseudo-cycle. In the second case we obtain a sliding pseudo-cycle topologically different from the
first one, or a tangent pseudo-cycle whose intersection with \( \Sigma \) is the tangency point \( p^+ \).

Now we assume \( p^- < p^+_1 < p^+ \). As before, in this case the Filippov vector field must connect \( p^+_1 \) to either \( p^- \) or \( p^+ \). In the first situation the analysis is trivial. In the second one there exist \( p^+_1 \leq p^+ \) the first point on \( \Sigma \) such that the positive trajectory of \( Y \) from \( p^+ \) meets \( \Sigma \). If \( p^+_1 < p^- \) then the positive trajectory of \( X \) connects \( p^+_1 \) to \( p^+_2 \in (p^+_1, p^+) \) since \( \Gamma_Z(t, p) \) visits \( \Sigma^e \) infinitely many times. In this case we obtain a sliding pseudo-cycle if \( p^- < p^+_2 < p^+ \) or a tangent pseudo-cycle if \( p^+_2 = p^+ \) (see Figure 6 (a)). On the other hand, if \( p^+_1 = p^- \) then we obtain a sliding pseudo-cycle (see Figure 6 (b)). If \( p^- < p^+_1 < p^+ \) then the Filippov vector field connects \( p^+_1 \) to either \( p^- \) or \( p^+ \) (see Figure 6 (c) for the first case) and we obtain a sliding pseudo-cycle. Finally, if \( p^+_1 = p^+ \) then we obtain a tangent pseudo-cycle whose intersection with \( \Sigma \) is the tangency point \( p^+ \) so we are done.

If \( p^+_1 = p^- \) we obtain a tangent pseudo-cycle whose intersection with \( \Sigma \) is the tangency point \( p^- \).

The statement (ii) is treated analogously to the previous case by reversing time in the trajectory \( \Gamma_Z(t, p) \). Finally, if statements (iii) or (iv) hold, then we have \( \Gamma_Z(t, p) \notin \Sigma^s \) for \( t > t_0 \) when \( \Gamma_Z(t_0, p) \in \Sigma^s \). In this situation the tangent-transversal case applies.

**Remark 4.** Notice that from the previous study several distinct sliding pseudo-cycles have emerged. For instance, some sliding pseudo-cycles we obtained before have only one sliding component whereas now the sliding pseudo-cycles may be contained in both regions \( \Sigma^+ \) and \( \Sigma^- \). Moreover, their intersections with \( \Sigma \) are composed by two components (cf. Figure 6). We also remark the appearance of new kind of tangent pseudo-cycle involving

**Figure 5:** The case \( p^+_1 \geq p^+ \).

**Figure 6:** The case \( p^- < p^+_1 < p^+ \) where the Filippov vector field connects \( p^+_1 \) to \( p^+ \). (a) \( p^+_1 < p^- \), (b) \( p^+_1 = p^- \) and (c) \( p^+_1 > p^- \) and Filippov vector field connects \( p^+_1 \) to \( p^- \).

**4.1.3 The tangent-tangent case assuming \( p^- = p^+ \)**

**Assertion 4.** Assume that hypotheses of Theorem 1 hold, \( X \) and \( Y \) have coincident tangency points on \( K \cap \Sigma \), \( p^- = p^+ \). Then \( \Gamma_Z(t, p) \) can be taken such that \( \omega(\Gamma_Z(t, p)) \) is a crossing pseudo-cycle.

**Proof.** Without loss of generality suppose that \( p^- = p^+ = 0 \). Now we get the following possible configurations:

(i) \( \Sigma^e = \mathbb{R} \setminus \{0\} \);

(ii) \( \Sigma^s = (-\infty, 0) \) and \( \Sigma^c = (0, +\infty) \);

(iii) \( \Sigma^c = (-\infty, 0) \) and \( \Sigma^s = (0, +\infty) \).

If statement (i) holds the crossing pseudo-cycles are obtained as in the Fundamental Lemma. So assume that statement (ii) holds. We claim that in this case there is no pseudo-cycle. Indeed let \( \Gamma \) be a closed maximal trajectory of \( Z \) such that \( \Gamma \cap \Sigma \neq \emptyset \). Assume that it does not contain neither equilibria nor pseudo-equilibria. We will show that \( \Gamma \) is neither positively nor negatively invariant and therefore it does not satisfies condition (i) of Definition 3.

Notice that the trajectories of the Filippov vector field \( Z_\Sigma \) are always increasing or decreasing in a small neighborhood of the double tangency \( (0, 0) \).
If \( \Gamma \cap \Sigma^a = \emptyset \) we have that \( \Gamma \) is not negatively invariant because \( \Gamma \) is a meager set while the saturation of a sliding segment in past time has non empty interior. Moreover, \( \Gamma \) is not positively invariant. In fact, if \( \Gamma \cap \Sigma^a \neq \emptyset \) then a argument similar to the previous is applicable. If \( \Gamma \cap \Sigma^e = \emptyset \) then 0 is a visible tangency point for at least on vector field \( X \) or \( Y \) and the Filippov vector field is increasing and therefore it is still increasing in \( \Sigma^e \) next to 0. Analogously we show that if \( \Gamma \cap e \neq \emptyset \) and therefore it is increasing in \( \Sigma \) \( \Gamma \) can be \( \tilde{\omega} \)-limit of a maximal trajectory \( \Gamma \) for some \( t > 0 \).

Remark 5. We notice that a crossing pseudo-cycle \( \Gamma \) can be \( \omega \)-limit of a maximal trajectory \( \Gamma_Z(t,p) \) that does not reaches \( \Gamma \) in finite time, according to Remark 4 in [3]. The same situation occurs for tangent pseudo-cycle \( \Gamma \). In fact, suppose that the visible tangency point \( p^+ \in \Gamma \) and consider the segment \( \Sigma = (p^+,p^+ + \epsilon) \) for \( \epsilon > 0 \). The claim is true if the first return map \( \tilde{P} : \Sigma \rightarrow \Sigma \) satisfies \(|P'(q)| < 1 \) for \( q \in \Sigma \). On the other hand, it does not occur for sliding pseudo-cycles. In fact, suppose that \( \Gamma \cap \Sigma^a \neq \emptyset \) and take \( q \in \Gamma \cap \Sigma^a \). There is a neighborhood \( V_q \) of \( q \) such that \( \Gamma_Z(t_0,p) \in V_q \) and every point of \( V_q \) reaches \( \Sigma^a \) in future finite time. So it follows that \( \Gamma_Z(t,\Gamma_Z(t_0,p)) \in \Gamma \) for some \( t > 0 \).

4.2 Mild Pseudo-cycles and Chaotic Sets

In this section we will explore the mild pseudo-cycles of type I, II and III, chaotic sets of type I, II and III and the relation between them for a planar Filippov vector field \( Z = (X,Y) \) as in Theorem 1.

Definition 4. A closed maximal trajectory \( \Gamma \) of a piecewise-smooth vector field \( Z \) such that \( \Gamma \cap \Sigma \neq \emptyset \) and it does not contain neither equilibria nor pseudo-equilibria is called a mild pseudo-cycle if \( \Gamma \) does not satisfies conditions (i) or (ii) in Definition 3, that is,

\( \neg (i) \) \( \Gamma \) is neither positively nor negatively invariant or
\( \neg (iii) \) \( \Gamma \) contains a proper maximal trajectory.

In particular we distinguish three types of mild pseudo-cycle: (a) mild pseudo-cycle of type I those satisfying only condition \( \neg (i) \) (see Figure 7 (a)), (b) mild pseudo-cycle of type II those satisfying only condition \( \neg (ii) \) (see Figure 7 (b)), and (c) mild pseudo-cycle of type III those satisfying both conditions \( \neg (i) \) and \( \neg (ii) \) (see Figure 7 (c)).

One of the well accepted definitions about chaotic sets is the one assuming topological transitivity, sensitive dependence on initial conditions and density of periodic orbits. The first three properties concerns Devaney’s conditions for the existence of chaos for smooth systems. Next we define chaos for PSVF.

Definition 5. A PSVF \( Z \) is topologically transitive on a compact invariant set \( \Lambda \) if for any pair of non-empty, open sets \( U \) and \( V \) in \( \Lambda \), there exist \( p \in U \) and \( \Gamma_Z(t,p) \) a positive global trajectory and \( t_0 > 0 \) such that \( \Gamma_Z(t_0,p) \in V \).

Definition 6. A PSVF \( Z \) exhibits sensitive dependence on a compact invariant set \( \Lambda \) if there exist a fixed \( r > 0 \) satisfying \( r < \text{diam}(\Lambda) \) such that for each \( x \in \Lambda \) and \( \epsilon > 0 \) there exist \( y \in B_\epsilon(x) \cap \Lambda \) and positive global trajectories \( \Gamma_Z^+(t,x) \) and \( \Gamma_Z^+(t,y) \) on \( \Lambda \) passing through \( x \) and \( y \), respectively, satisfying

\[ d(\Gamma_Z^+(t,x),\Gamma_Z^+(t,y)) > r \quad \text{for some } t > 0, \]

where \( d \) is the the Euclidian distance in \( \mathbb{R}^2 \).

Definition 7. We say that a compact invariant set \( \Lambda \) is chaotic for a piecewise-smooth vector field \( Z \) if

(a) \( Z \) is topologically transitive on \( \Lambda \);
(b) \( Z \) exhibits sensitive dependence on \( \Lambda \) and
(c) the periodic orbits of \( Z \) are dense on \( \Lambda \).
Moreover, we distinguish three situations: (i) Λ is chaotic of type I if it has empty interior and does not present sliding motion; (ii) Λ is chaotic of type II if it has empty interior presenting sliding motion and (iii) Λ is chaotic of type III if it has non-empty interior, see Figure 8.

**Proposition 1.** Every mild pseudo-cycle of type II or III is chaotic of type I or II.

**Proof.** Let Γ be a mild pseudo-cycle of type I or II, that is, (i) Γ is a closed maximal trajectory of Z, (ii) Γ ∩ Σ ≠ ∅, (iii) Γ does not contain neither equilibria nor pseudo-equilibria and (iv) Γ contains a proper maximal trajectory.

Since (i) holds we have that int(Γ) = ∅. Now we must show that (a) Z is topologically transitive on Γ, (b) Z exhibits sensitive dependence on Γ and (c) the periodic orbits of Z are dense.

We see that being Γ itself a closed maximal trajectory, we have that (c) holds. To prove (a) and (b) let ̃Γ ⊂ Γ be a proper maximal trajectory and consider p ∈ ̃Γ a point such that there is no uniqueness of trajectory in Γ, so (a) holds. Moreover, let γ1(t, p) = Γ and γ2(t, p) = ̃Γ be these maximal trajectories, with γ1(0, p) = γ2(0, p) = p. Then, we have that 2r = supγ1>0 d(γ1(t, p), γ2(t, p)) > 0. Thus, given x ∈ Γ if y ∈ Γ is sufficiently close to x we have that ΓZ(t, p) = p = ΓZ(p) for t sufficiently close to t. Therefore we obtain the sensitive dependence in Γ from d(γ1(t, p), γ2(t, p)) > r for some t > 0.

**Assertion 5.** Assume that hypotheses of Theorem 1 hold and X and Y have coincident tangency points p− and p+ on K ∩ Σ. Then ΓZ(t, p) can be taken such that ω(ΓZ(t, p)) is a mild pseudo-cycle of type I, II or III.

**Proof.** Without loss of generality suppose that p− = p+ = (0, 0). So we have the following possible configurations:

(i) Σ− = R \ {0};

(ii) Σ+ = (−∞, 0) and Σ− = (0, +∞);

(iii) Σ+ = (−∞, 0) and Σ− = (0, +∞).

Assume that statement (i) holds. Then according to Fundamental Lemma we obtain a mild pseudo-cycle of type II as ω−limit. Indeed this is the case where (0, 0) is a hyperbolic double tangency. Moreover ΓZ(t, p) is the union of two periodic orbits of X and Y both tangent to Σ at (0, 0), see Figure 8 (c).

Now, assume that statement (ii) holds. From Remark 1 so we only consider infinitely many transition from and to Σ− ∪ Σ+ = Σ. Under such assumption, we claim that the double tangency point p− = p+ = (0, 0) cannot be elliptic. Indeed, if the tangency is elliptic and since Σ− ∩ Σ+ = ∅ then either (0, 0) is an equilibrium of Y or X, and in this case it is an equilibrium of the extended Filippov vector field (cf. Remark 2), or (0, 0) attracts the Filippov vector field in the escaping region and repels it in the sliding region, by continuity of X and Y. In both cases Σ is invariant for the extended Filippov vector field so transitions cannot occur. Therefore the double tangency is parabolic or hyperbolic. Moreover, notice that the Filippov vector field is always increasing or decreasing in a neighborhood of the double tangency (0, 0). Let ΓZ(t, p) be a maximal trajectory of Z contained on a compact set K. We split the analysis into the following cases:

(1) ΓZ(t, p) visits Σ+ ∪ Σ− a finite number of times;

(2) ΓZ(t, p) visits Σ+ a finite number of times and visits Σ− an infinite number of times;

(3) ΓZ(t, p) visits Σ+ an infinite number of times and visits Σ− a finite number of times;

(4) ΓZ(t, p) visits both Σ+ and Σ− an infinite number of times.

Suppose that situation (1) occurs. Since there is infinitely many transitions of ΓZ(t, p) from and to Σ, then there exist t0 > 0 such that ΓZ(t, p) ∩ Σ = (0, 0) for t > t0 and ΓZ(t, p) with t > t0 is either a periodic orbit of X or Y tangent to Σ on (0, 0) or an union of both such periodic orbits. Since the Filippov vector field is increasing or decreasing in a neighborhood of (0, 0), in the first case we obtain a mild pseudo-cycle of type I and in the second case we obtain a mild pseudo-cycle of type III.

Now, suppose that situation (2) occurs. Then there is t0 > 0 such that ΓZ(t, p) ∉ Σ− for t > t0. Since it occurs infinitely many transitions of ΓZ(t, p) from and to Σ, we have that the returns of ΓZ(t, p) to Σ with t > t0 occur through a visible
tangency \((0,0)\). If \(\Gamma_Z(t,p)\) returns to \((0,0)\) by \(X\) or \(Y\) there is \(q_1^+ \in \Sigma^c\) the point for which the past of \((0,0)\) by \(X\) or \(Y\) meets \(\Sigma\). Moreover, at least one of the points \(q_1^-\) or \(q_1^+\) belong to \(\Sigma^c\) and, for \(t > t_0\) we have that \(\Gamma_Z(t,p)\) leaves \(\Sigma^c\) only by \(q_1^-\) according to \(X\) or by \(q_1^+\) according to \(Y\), see Figures 8 (a) and (b). Once the Filippov vector field is increasing in a neighborhood of \((0,0)\) in this scenario, we obtain a mild pseudo-cycle of type I if there exist only one of \(q_1^-\) or \(q_1^+\) and a mild pseudo-cycle of type III if there exists both \(q_1^-\) and \(q_1^+\). The case (3) is analogous by reversing orientation on time.

Finally, suppose that situation (4) occurs and notice that \(K \cap \Sigma\) is a compact subset of \(\Sigma\). Suppose also that there exist \(\tilde{p}_s \in \Sigma^s\) and \(\tilde{p}_c \in \Sigma^c\) the pseudo-equilibria nearest to \((0,0)\). Let \(p'_s\) and \(p'_c\) be the frontier of \(K \cap \Sigma\) in \(\Sigma^s\) and of \(K \cap \Sigma\) in \(\Sigma^c\), respectively. Again we assume that these points are the nearest to \((0,0)\). Put \(p_s = \max\{\tilde{p}_s, p'_s\}\) and \(p_c = \max\{\tilde{p}_c, p'_c\}\). If there is no pseudo-equilibrium on \(\Sigma^s\) or \(\Sigma^c\), then \(p_s = p'_s\) and \(p_c = p'_c\), respectively. We get

- \(\Gamma_Z(t,p) \cap \Sigma^s \subset \{p_s, 0\}\), reaching \(p_s\) only if \(p_s = p'_s\);
- \(\Gamma_Z(t,p) \cap \Sigma^c \subset \{0, p_c\}\), reaching \(p_c\) only if \(p_c = p'_c\);
- The extended Filippov vector field connects \(p_s\) to \(p_c\).

In this way we have that \(\Gamma_Z(t,p)\) is a maximal closed trajectory that is not positively nor negatively invariant and therefore \(\Gamma_Z(t,p)\) is a mild pseudo-cycle of type II or III.

The statement (iii) is verified in an entirely analogous way. \(\square\)

### 4.3 Chaotic set of type III and pseudo-graphs

In this section we study the relation between chaotic sets of type III and those pseudo-graphs that are \(\omega\)-limit of some maximal trajectory in the sense of Theorem 1.

**Definition 8.** Let \(Z\) be a piecewise-smooth vector field. A closed curve \(\Gamma\) is a pseudo-graph if \(\Gamma \cap \Sigma \neq \emptyset\) and it is an union of trajectory-arcs of \(Z\) joining equilibrium or pseudo-equilibrium (see Figure 9).

Suppose that \(\Sigma^s \cup \Sigma^c \neq \emptyset\). Let \(\Omega_g\) be the set of pseudo-graphs and \(\Omega^s \subset \Omega_g\) be the subset of pseudo-graphs that are \(\omega\)-limit of some maximal trajectory of \(Z\). That distinction is necessary because some pseudo-graphs are not \(w\)-limit of any trajectory.

Next result provides a partial description of the elements of \(\Omega^s\).

**Proposition 2.** Assume that hypotheses of Theorem 1 hold and let \(\Gamma \in \Omega^s\) such that \(\Gamma \cap (\Sigma^s \cup \Sigma^c) \neq \emptyset\), that is, \(\Gamma\) is a pseudo-graph which is the \(\omega\)-limit of a maximal trajectory of \(Z\) with sliding motion. Then, \(X\) and \(Y\) have coincident tangency points \(p^- = p^+\).

**Proof.** Suppose initially that there exist \(q \in \Gamma \cap \Sigma^s\). Since \(\Gamma = \omega(\Gamma_Z(t,p))\) there exist a sequence \(p_n = \Gamma_Z(t_n, p)\) converging to \(q\), where \(t_n \to +\infty\) when \(n \to +\infty\). Notice that there exist a neighborhood \(V_q\) of \(q\) such that every point in \(V_q\) meets \(\Sigma^s\) in finite time. So take \(q_n = \Gamma_Z(t_n, p_n) \in \Sigma^s\) with \(t_n > 0\). Once \(p_n \to q\) we have that \(q_n \to q\). Moreover, the future of all \(q_n\) for \(n\) sufficiently large visits a visible tangency point, that we suppose without loss of generality to be \(p^-\). So it follows that there are no pseudo-equilibria between \(q\) and \(p^-\). If \(p^- \neq p^+\) then we have that \(\Sigma^s\) does not
connect with \( \Sigma^c \) and the motion of \( \Gamma_Z(t,p) \) is regular periodic whose \( \omega \)-limit is itself and therefore \( \omega(\Gamma_Z(t,p)) \neq \Gamma \). An analogue contradiction is obtained in the regular-tangent case. So \( p^- = p^+ \) and in the regular-tangent and \( p^- \neq p^+ \) cases a pseudo-graph that intersects \( \Sigma^c \) cannot be the \( \omega \)-limit of a maximal trajectory. Analogously we get the same result for a pseudo-graph \( \Gamma \) intersecting \( \Sigma^c \) by reversing time in the previous analysis.

The next Assertion is a partial converse of the last result.

**Assertion 6.** Assume that hypotheses of Theorem 1 hold and \( X \) and \( Y \) have coincident tangency points \( p^- \) and \( p^+ \) on \( K \cap \Sigma \). Then \( \Gamma_Z(t,p) \) can be taken such that \( \omega(\Gamma_Z(t,p)) \) is a chaotic set \( \Lambda \) of type III. Moreover, if \( \mathcal{G} = \partial \Lambda \cup [\Lambda \cap \Sigma] \) contains any equilibrium or pseudo-equilibrium, then \( \mathcal{G} \) is a pseudo-graph and \( \Gamma_Z(t,p) \) can be taken such that \( \omega(\Gamma_Z(t,p)) \neq \mathcal{G} \).

We initially consider the parabolic case and we split the proof of previous Assertion in three parts: 1) We construct the set \( \Lambda \); 2) we show that \( \Lambda \) is the \( \omega \)-limit set of a maximal trajectory and also that there is another maximal trajectory in such way that \( \mathcal{G} \) is its \( \omega \)-limit and 3) we show that \( \Lambda \) is a chaotic set of type III. Without loss of generality suppose that \( (0,0) \) is visible for \( X \) and invisible for \( Y \). Moreover take \( p_s \) and \( p_e \) as in the proof of Assertion 5 and let \( q^+_e \in [0,p_e] \) be determined in the following way, see Figure 10:

- If \( q \in (0,q^+_e) \) then the future of \( q \) through \( Y \) meets \( [p_s,0] \) at \( q^+ \) and it does not occur if we consider another initial condition \( q > q^+_e \) in \( \Sigma^c \) (unless possibly \( q^+_e = p_e \));
- for each \( q \in (0,q^+_e) \) there is no equilibrium point of \( Y \) on the region whose border is the arc of trajectory of \( Y \) in the previous bucket with end points \( q \) and \( q^+ \in [p_s,0) \) and the segment \([q^+,q] \subset \Sigma \).

The portion of \( \Lambda \) contained on \( \Sigma^- \) is the closure of the union of the regions described in the last bucket. We now describe the portion of \( \Lambda \) contained on \( \Sigma^+ \). Indeed, if neither the past nor the future of the tangency point \((0,0)\) through \( X \) meet \([p_s,p_e]\) then \( \Lambda \cap \Sigma^+ = \{q^+_s,q^+e\} \subset \Sigma \). Then suppose that the past of \((0,0)\) by \( X \) meets \((0,p_e)\) in \( q^+_1 \) and let \( q^-_e \in [q^-_1,p_e] \subset \Sigma^c \) be defined as follows, see Figure 10 (b):

- if \( q \in (q^-_1,q^-_e) \) then the future of \( q \) through \( X \) meets \([p_s,0] \) at \( q^- \). Moreover, it does not occur for another initial condition \( q > q^-_e \) in \( \Sigma^c \) (unless possibly if \( q^-_e = p_e \));
- for each \( q \in [q^-_1,q^-_e] \) there is no equilibrium point of \( X \) in the region whose border are the arcs of trajectories \( \gamma_1 \) and \( \gamma_2 \) of \( X \), where \( \gamma_1 \) has end point \((0,0)\) and \( q^-_1 \) and \( \gamma_2 \) has boundaries \( q \) and \( q^- \) plus the segment \([q^-_1,q] \subset \Sigma \).

The portion of \( \Lambda \) contained on \( \Sigma^- \) is the closure of the union of the regions described in the last bullet. The case where the future of \((0,0)\) by \( X \) meets \([p_s,0] \) is treated in similar way and it is expressed in Figure 10 (a).

In case the double tangency point if of hyperbolic type we treat analogously to get \( q^- \) through the vector field \( X \) and we also obtain \( q^+_1 \) through the vector field \( Y \) with entirely analogous properties.

We now prove the second part. Indeed, from the previous construction we see that \((0,0)\) is the common boundary between the sliding and the escaping region. Moreover, for any interior points \( r,s \) of \( \Lambda \) there exists \( t_r > 0, t_s < 0 \) and trajectories \( \Gamma^r_2 \) and \( \Gamma^s_2 \) such that \( \Gamma^r_2(t_r,r) = \Gamma^s_2(t_s,s) = (0,0) \). Hence any two points of the interior can be connected by an arc of some maximal trajectory.

Therefore the interior of \( \Lambda \) is the \( \omega \)-limit set of a maximal trajectory.

Now, consider an arbitrary point \( p \in \Lambda \) and two sequences \( \{p_n\} \subset (0,q^+_s) \) and \( \{q_n\} \subset (0,q^-_e) \) converging to \( q^+_s \) and \( q^-_e \), respectively. Consider also
the maximal trajectory $\Gamma_Z(t,p)$ given by the following way. Starting at $p$, let the trajectory reach $(0,0)$. From the previous comment about recurrence through $(0,0)$, there exists two sequences $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < \cdots$ with $t_n \to \infty$ such that $\Gamma_Z(s_n, p) = p_n$, $\Gamma_Z(t_n, p) = q_n$. Therefore $\Gamma_Z(t, p)$ leaves $\Sigma^e$ by $Y$ at $p_n$ for $t = s_n$. By continuity of trajectories respect to initial conditions and using the compactness of $\Lambda$ it follows that $\Gamma_Z(t, p)$ can be taken such that $\omega(\Gamma_Z(t, p)) = G \subset \partial \Lambda \subset \Lambda$. Moreover, if $G$ contains no equilib-rium or pseudo equilibrium we are done. Otherwise it is a pseudo-graph so the second part of the statement is proved.

The following result finishes the proof of Assertion 6.

**Proposition 3.** The set $\Lambda$ constructed above is chaotic of type III.

**Proof.** Let $U$ and $W$ be open non-empty subsets of $\Lambda$. For a point $p \in U$ there exist a positive time $t_0 > 0$ such that $\Gamma_Z(t_0, p) = 0$. Since the positive trajectories of $Z$ contained on $\Lambda$ starting at 0 are dense on $\Lambda$ by construction, there exist $t_1 > 0$ such that $\Gamma_Z(t_1, 0) \in W$. Then $\Gamma_Z(t_1 + t_1, p) \in W$. Therefore $\Lambda$ is topologically transitive. Analogously we show the density of periodic orbits on $\Lambda$. For the sensitive dependence of $Z$ on $int(\Lambda)$, we noticed that for $x$ sufficiently close to $y$ on $int(\Lambda)$, there exist a time $t > 0$ sufficiently close to $s > 0$ such that $\Gamma_Z(t, x) = \Gamma_Z(s, y) = p_0$. If $r = \frac{1}{2}diam(\Lambda)$ then there exists trajectories $\Gamma_Z(t, x)$ and $\Gamma_Z(t, y)$ of $Z$ such that $d(\Gamma_Z(T, x), \Gamma_Z(T, y)) > r$ for some $T > 0$. Moreover, clearly the interior of $\Lambda$ is non-empty.

**Remark 6.** We see that $\Lambda$ cannot be a minimal set since it is either non-invariant or there always exist a non-empty invariant proper subset.

As an immediate consequence of the previous discussion we have the following result:

**Proposition 4.** Assume that hypothesis of Theorem 1 hold and that $Z$ is chaotic on the compact set $K$. Then,

(i) there exist a parabolic or hyperbolic (double) tangency point on $K$;

(ii) $\Sigma^c = \emptyset$;

(iii) there exist a maximal trajectory $\Gamma_Z(t, p)$ of $Z$ visiting and living $\Sigma^c$ and $\Sigma^s$ infinitely many times.

In particular, Proposition 4 states that, under the hypotheses of Theorem 1, if a PSVF is chaotic then it is structurally unstable. It holds due to the connection of tangency points on $\Sigma$ which is clearly broken by suitable small perturbations.

5 Proof of the main results

Now we synthesize the previous analysis in order to establish the proof of the main results of the paper.

**Proof of Fundamental Lemma.** First assume that $\Gamma_Z(t, p) \cap \Sigma^c \cup \Sigma^s \neq \emptyset$. In this case, by using the proof provided in [3] we get that $\omega(\Gamma_Z(t, p))$ is one of items from (i) to (vii).

Assume now that last situation occurs only after a finite time $t_0 \in \mathbb{R}$, that is, $\Gamma_Z(t, p) \notin \Sigma^c \cup \Sigma^s$, $\forall t > t_0$. Since we are interested in limit sets we only need to consider $t \in \mathbb{R}$ arbitrarily large, therefore we also get items (i) to (vii) in the last situation.

On the other hand, given a fixed $T \in \mathbb{R}$, if $\Gamma_Z(t, p) \in \Sigma^c \cup \Sigma^s$, $\forall t > T$ and since $\Gamma_Z^+(t, p)$ is contained on a compact set it follows that $\omega(\Gamma_Z(t, p))$ must accumulate in a point on $q \in \Sigma^c \cup \Sigma^s$. If such a point belongs to $\Sigma^c \cup \Sigma^s$ then it is a pseudo-equilibrium, so we get statement (viii). If $q$ belongs to the boundary of $\Sigma^c \cup \Sigma^s$ then $q$ must be a tangency point. From that point we could concate-enate other trajectories of $X$ or $Y$ to $q$ unless $q$ is an invisible tangency for a vector field and a visible tangency of odd order to another, being $q$ reached in finite positive time in the last situation. Under this configuration if $q$ is an attractor point to the sliding segment, then it is a tangency of type II. Therefore we have statement (ix). If $q$ is a repeller so by the previous argument it should accumulate in a pseudo-equilibrium point and we are done.

**Proof of Theorem 1.** Let $K \subset \mathbb{R}^2$ be a nonempty compact subset of $\mathbb{R}^2$ and take $p \in K$ such that the positive maximal trajectory $\Gamma_Z^+(t, p)$ is contained on $K$. First assume that $\Gamma_Z(t, p) \notin \Sigma^c \cup \Sigma^s$ for $t$ sufficiently large. If the orbit does not touches $\Sigma$ again then according to Fundamental Lemma
\( \omega(\Gamma_Z(t,p)) \) is either an equilibrium, a periodic orbit or a graph of \( Y \) or \( X \) so we get items (i), (ii) and (iii). Otherwise \( \omega(\Gamma_Z(t,p)) \) is either a crossing pseudo-cycle, a pseudo-graph, a mild pseudo cycle of type II or a tangency of type I of \( Z \), so now we get (partially) items (v) – (viii). On the other hand, if \( \Gamma_Z(t,p) \in \Sigma^+ \cup \Sigma^- \) for \( t \) sufficiently large then again according to Fundamental Lemma \( \omega(\Gamma_Z(t,p)) \) is either a pseudo-equilibrium of \( Z \) or a tangency of type II so we get items (iv) and (viii).

Suppose now that for any \( T > 0 \) there exists \( s,t > T \) such that \( \Gamma_Z(s,p) \in \Sigma^+ \cup \Sigma^- \) and \( \Gamma_Z(t,p) \notin \Sigma \). Notice that under this assumption (a) the existence of at least one tangency point is guaranteed on \( \Sigma \cap K \) for \( X \) or \( Y \) or both and (b) when there exist two distinct tangency points they cannot be simultaneously invisible (see Subsection 4.1.2). Moreover, by hypothesis these vector fields have at most one tangency point on \( \Sigma \cap K \). Then analyzing \( X \) and \( Y \) on \( \Sigma \cap K \) we obtain four different situations:

- **1)** \( X \) has a tangency point and \( Y \) is regular (or reciprocally). In this case by Assertion 1 we have that \( \omega(\Gamma_Z(t,p)) \) is a tangent or a sliding pseudo-cycle contained entirely in \( \Sigma^+ \) or \( \Sigma^- \) so we get item (v) of the theorem.

- **2)** \( X \) and \( Y \) have non coincident tangency points of opposite visibility. In this case by Assertion 2 the \( \omega(\Gamma_Z(t,p)) \) is either a crossing or tangent or sliding pseudo-cycle then we get item (v).

- **3)** \( X \) and \( Y \) have non coincident visible tangency points. In this case by Assertion 3 the \( \omega(\Gamma_Z(t,p)) \) is either a crossing or tangent or sliding pseudo-cycle therefore we get item (v).

- **4)** \( X \) and \( Y \) have coincident tangency points. In this case the \( \omega(\Gamma_Z(t,p)) \) is either a crossing pseudo-cycle (Assertion 4) or a mild pseudo-cycle of type I, II or III (Assertion 5) or a chaotic set of type III or it is a pseudo-graph of \( Z \) (Assertion 6). In these cases we get items (vi), (vii), (ix) and (viii), respectively. The four previous cases contemplate all the objects listed in Theorem 1 and so the proof is ended.

As commented in Section 2 the pseudo-cycles and pseudo-graphs listed in Theorem 1 may have different topological types. Indeed, the pseudo-cycles obtained in case 1) of the proof are contained on \( \Sigma^+ \) or \( \Sigma^- \) and their intersection with \( \Sigma \) have only one component. On the other hand, the pseudo-cycles obtained in case 2) are contained on either \( \Sigma^+ \) or \( \Sigma^- \) or in both \( \Sigma^\pm \) and their intersection with \( \Sigma \) has again one component. The pseudo-cycles obtained in case 3) of the proof are contained on \( \Sigma^+ \) or \( \Sigma^- \) or in both \( \Sigma^\pm \). However, now the intersection of the pseudo-cycle with \( \Sigma \) has one or two components. Concerning the pseudo-graphs, those obtained from Fundamental Lemma have neither sliding nor escaping segments while the pseudo-graphs from case 4) have both type of regions simultaneously. In any case, it is easy to see that those objects have distinct topological types.

**Proof of Theorem 2.** By subsection 4.3 there is a non-minimal chaotic set \( \Lambda \) of type III for \( Z \). Given \( p \in int(\Lambda) \) there are \( t_0, t_1 > 0 \) such that \( \Gamma_Z(t_0,p) = p_0 \) and \( \Gamma_Z(t_1,p_0) = p \). Then every point in \( int(\Lambda) \) is periodic. In particular the periodic orbits of \( Z \) are dense in \( \Lambda \). Moreover, since there exist a dense trajectory in \( \Lambda \) passing through \( p_0 \) it follows that there exist a dense trajectory in \( \Lambda \) passing through \( p \in int(\Lambda) \). Since \( \Gamma_Z(t_0,p) = p_0 \) with \( t_0 > 0 \) it also follows that \( \omega(\Gamma_Z(t,p)) = \omega(\Gamma_Z(t_0,p)) = \Lambda \) and therefore \( p \in int(\Lambda) \) is a non-trivial recurrence point. Finally, for all \( V \) neighborhood of \( p \) and \( t_0 > 0 \) there exist \( t > t_0 \) such that \( \Gamma_Z(t,p) \in V \), where \( \Gamma_Z(t,p) \) is a dense trajectory on \( \Lambda \). Therefore, \( p \) is a non-trivial non wandering point.

**6 Applications to piecewise-linear vector fields separated by straight lines**

Piecewise-linear vector fields have been widely studied due to its theoretical and practical importance. Moreover, this class of Filippov systems is particular interesting because one can easily find its solutions. Regardless, in what follows we see that piecewise-linear vector fields not only verify the conditions of Theorem 1 but the \( \omega \)-limit set of a maximal trajectory may be a chaotic set of type III.

Let \( Z = (X,Y) \) be a planar piecewise-linear vector field with \( \Sigma = \{(0,x_2) : \ x_2 \in \mathbb{R} \} \) and \( X_{\pm}(x_1,x_2) = A^\pm(x_1,x_2)^T + b^\pm \), being

\[
A^\pm = \begin{pmatrix} a_{11}^\pm & a_{12}^\pm \\ a_{21}^\pm & a_{22}^\pm \end{pmatrix} \quad \text{and} \quad b^\pm = (b_1^\pm, b_2^\pm)^T.
\]
Assume that $\det(A^\pm) \neq 0$ so $X$ or $Y$ have only one equilibrium which we assume to be located outside $\Sigma$. Note that these are generic assumptions. Also, notice that for $p = (0, x_2) \in \Sigma$ we have $X_+ f(p) = a_{12}^+ x_2 + b_1^+$, and $X_2^+ f(p) = a_{22}^+ (a_{12}^+ x_2 + b_2^+)$. Consequently there exist at most one tangency point for $X$ or $Y$ at $p^\pm = (0, -b_2^+/a_{12}^+)$ if $a_{12}^+ \neq 0$. It is easy to check that $p^\pm$ is either a fold point of $X$ or $Y$ or a tangency point of infinite order. However, the second situation in equivalent to the existence of an equilibrium point on $\Sigma$ which we are avoiding here. Moreover the pseudo-equilibria are isolated.

From the previous remarks we see that $Z$ satisfies the hypotheses of Theorem 1. Therefore the $\omega$-limit set of a maximal trajectory $\Gamma_Z(t, p)$ contained on a compact set is one of the possibilities from (i) to (viii) of Theorem 1. However, we stress that a graph cannot occurs because there is not such objects in linear vector fields.

Next we present some particular piecewise-linear vector fields.

### 6.1 Relay systems

The class of relay systems is very important in areas as control theory and in friction phenomena (see for instance [1], [7] and [9]). According to [7] a planar relay system is a piecewise-smooth vector field that can be written in the general form

$$
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= C^T x, \\
u &= -\text{sgn}(y),
\end{align*}
$$

where $A$, $B$ and $C$ are real matrices. Therefore the relay system are indeed piecewise-linear vector fields. The sign function induces a separation region which in this case is again a straight line. So as observed previously these systems satisfy the conditions of Theorem 1.

### 6.2 Existence of chaotic sets of type III

Now we introduce a piecewise-linear vector field $Z$ presenting a chaotic system of type III. In order to do that, consider $Z = (X, Y)$ separated by the straight line $\Sigma = \{(0, y) : y \in \mathbb{R}\}$, where

$$X(x, y) = \left(-\frac{1}{2} x - y - 1, x + \frac{1}{2} y + 2\right)$$

and

$$Y(x, y) = (x + y + 1, -2x - y - 2).$$

The equilibrium points are centers located at $(-2, 0)$ for $X$ and at $(1, 0)$ for $Y$. Setting $\Sigma = f^{-1}(0)$ with $f(x, y) = x$ we get $Xf(0, y) = -y - 1$ and $Y(0, y) = y + 1$. So both vector fields $X$ and $Y$ have tangency points at $(0, -1)$ which is visible for $X$ and invisible for $Y$. Since $Xf(0, 1) < 0$ and $Yf(0, 1) > 0$ it follows that $\Sigma^e = (-1, +\infty)$ and $\Sigma^a = (-\infty, -1)$. A simple calculation shows that the Filippov vector field is given by \( \dot{x} = 0, \dot{y} = -y/4 \). Therefore $(0,0)$ is the unique pseudo-equilibrium point. Moreover, it can be extended beyond the tangency point $(0, -1)$ which in this case become an attractor equilibrium point for the extended Filippov system (see Figure 11). Now we construct the chaotic set of type III as follow: let $\tilde{\Lambda} \subset \mathbb{R}^2$ be the subset delimited by the following curves

$$
\begin{align*}
\Gamma_1 : \text{the orbit of } X \text{ connecting the pseudo-equilibrium } (0, 0) \text{ to } \Sigma^a; \\
\Gamma_2 : \text{the orbit of } Y \text{ connecting } (0, 0) \text{ to } \Sigma^a; \\
\Gamma_3 : \text{the periodic orbit passing through the tangency point } (0, -1) \text{ according to } X.
\end{align*}
$$

That lead us to the following.

**Proposition 5.** $\tilde{\Lambda}$ is a chaotic set of type III and it is a positively minimal set.

**Proof.** The fact that $\tilde{\Lambda}$ is a chaotic set of type III follows from the proof of Proposition 3. Besides, $\tilde{\Lambda}$ is clearly a non empty, compact and positively invariant set. In order to see that $\tilde{\Lambda}$ is indeed minimal, we see that if $p \in \text{int}(\Lambda)$ then no positive
maximal trajectory \( \Gamma_2^s(t,p) \) reaches the pseudo-equilibrium \((0,0)\) and therefore \( \Gamma_2^s(t_0,p) \) is contained on the interior of \( \Lambda \). On the other hand, if \( q \in \partial \Lambda \) then the saturation of \( q \) in future time is the whole \( \Lambda \) except possibly by a segment on its boundary. Thus \( \Lambda \) cannot contain any non empty, compact and positively invariant proper subset, that is, \( \Lambda \) is a positively minimal set.

The discussed example shows that some linear behavior may lead to chaos, which may not be desired in some situations as those related to real world applications. However, in the linear scenario, we can provide a more detailed version of Proposition 4 in terms of the parameters of the system. Indeed, with the notations introduced at the beginning of the current section and using Corollary 6 we have the following result.

**Proposition 6.** If \( Z \) has chaotic behavior then the following statements hold (simultaneously):

(i) \( Y \) and \( X \) have coincident tangency points, that is, \( a_{12}^1b_1^+ - a_{12}^1b_1^- = 0 \);

(ii) the double tangency is parabolic or hyperbolic, that is, \( a_{12}^2b_2^- - a_{22}^2b_2^+ < 0 \) or \( a_{12}^2b_2^+ - a_{22}^2b_2^- > 0 \);

(iii) \( \Sigma^c = \emptyset \), that is, \( a_{12}^1a_{12}^- < 0 \).

The former result is important because it provides necessary conditions to the existence of chaos in piecewise-linear vector field. For instance, it states that such a system having sewing points cannot present chaos.

### 6.3 The case of three zones

Next we present an example to stress out the importance of the number of regions in the hypotheses of Theorem 1. We illustrate that the following piecewise-linear vector field with three linearity zones

\[
Z(x,y) = \begin{cases} 
(-y - 1, x) & \text{if } \|(x,y)\| \geq 1, \\
(-2y, x) & \text{if } \|(x,y)\| \leq 1. 
\end{cases} \tag{4}
\]

We have two lines of discontinuity given by \( \Sigma_1 = \{x = -1\} \) and \( \Sigma_2 = \{x = 1\} \), so we call \( \Sigma = \Sigma_1 \cup \Sigma_2 \). Moreover, it is easy to see that \( \Sigma \) can be split into intervals of the form \( \Sigma_1' = (-\infty, -1) \cup (0, +\infty) \), \( \Sigma_1 = (-1, 0) \), \( \Sigma_2 = (-\infty, -1) \cup (0, +\infty) \) and \( \Sigma_2 = (-1, 0) \). The Filippov vector field on \( \Sigma_1 \) and \( \Sigma_2 \) is given by \( Z^e(x,y) = (0, x) \).

Let \( \Lambda \) be the set on \( \mathbb{R}^2 \) delimited by the curves \( \Gamma_i, i = 1, \ldots, 5 \), where \( \Gamma_1, \Gamma_3 \) and \( \Gamma_5 \) are arc of trajectories of \( Y \) connecting \((-1,0)\) to \((1,0)\), \((-1,-2)\) to \((1,-2)\) and \((1,0)\) to \((-1,0)\), respectively; \( \Gamma_2 \) and \( \Gamma_4 \) are arc of trajectories of \( X \) connecting \((-1,0)\) to \((-1,-2)\) and \((1,-2)\) to \((1,0)\), respectively (see Figure 12).

The following results describe the behavior on \( \Lambda \).

**Proposition 7.** The non-empty set \( \Delta \) satisfies the following properties:

1. it is compact and invariant;
2. it does not contain proper subsets verifying property (1);
3. it is topologically transitive;

Moreover, the system (4) exhibits sensitive dependence on \( \Lambda \).

**Proof.** Statement (1) follows from the construction of \( \Lambda \). Moreover, notice that for every point \( p \in \Lambda \) there exist a \( t > 0 \) such that \( \Gamma_Z(t,p) = (-1,0) \) and for each \( q \in \Lambda \) there exist a \( s > 0 \) such that \( \Gamma_Z(s,(-1,0)) = q \). Then for each \( p, q \in \Lambda \) there exist a value \( s + t > 0 \) such that \( \Gamma_Z(s+t,p) = q \). That proves properties (2) and (3).

Now take \( x \in \Lambda \) and fix \( r = (1/2)\text{diam} \Lambda \). If \( y \in \Lambda \) is sufficiently close to \( x \) then there exist \( s \) sufficiently close to \( s \) such that \( \Gamma_Z(t,y) = \Gamma_Z(s,x) = (-1,0) \). The Filippov vector field on \( \Sigma_1 \) and \( \Sigma_2 \) is given by \( Z^e(x,y) = (0, x) \).
Then there exist positive global trajectories $\Gamma^+\!_x$ and $\Gamma^+\!_y$ passing through $x$ and $y$, respectively, satisfying $d(\Gamma^+\!_x(T),\Gamma^+\!_y(T)) > r$ for some $T > 0$. It finishes the proof of the proposition. 

The last result states that $\Delta$ is a minimal set which cannot be achieved in Theorem 1. As we commented before, it happens because the number of regions defining the PSVF plays an important role in the shape of the limit sets.

7 Conclusions

In this paper we studied some objects emerging from the theory of PSVF as $\omega$-limit of a maximal trajectory $\Gamma_Z(t,p)$. In particular we introduce new objects which actually are a sophistication of some known concepts. More precisely, we distinguish pseudo-cycles from mild pseudo-cycles and we study three different types chaotic sets. Such a refinement is necessary once structural unstable situations may occur as $\omega$-limit even in simple contexts as the linear one. In particular for the linear case, we provide some classes for which the main result of the paper applies.

We also observed the absence of nontrivial minimal sets under the hypotheses of Theorem 1 because generally invariance cannot be guaranteed unless we assume (i) more tangency points (consequently more sliding and escaping regions) or (ii) more regions defining the PSVF. Besides that, we identified the existence of orientable minimality and chaos.

It is important to highlight that the global study performed throughout this paper allow us to identify not only $\omega-$limit sets but also the own structure of trajectories in a more embracing scenario. For instance, besides the facts concerning minimal sets, chaos and transitivity, we detect the existence of homoclinic and heteroclinic connections, several different kind of limit sets (crossing, tangential or having sliding) as well new objects as tangency points of type II where trajectories may also accumulate.

Finally we call the attention to a potential application of the global analyzes performed in this paper addressing bifurcation theory. Indeed, let $Z = (X,Y)$ be a PSVF such that each vector fields $X$ and $Y$ contribute with one tangency point of even order, $p^{-}$ and $p^{+}$ with $p^{-} \neq p^{+}$. Assume that these tangency points are connected through a maximal trajectory. This trajectory is a tangential pseudo cycle having two tangent points on it and a pseudo-equilibrium of saddle type inside (see Figure 13). One can verify the bifurcation of sliding limit cycles of different topological types by breaking the fold connections. Indeed, the limit cycles may occupy one or two zones as well as they can be formed by one or two arcs of sliding. The same occurs in other configurations studied in the paper. To understand such bifurcations and to provide their unfolding is a hard task that have been under explored in the literature despite the exhaustive number of phenomena modeled by PSVF.

References


