

Minimal sets and chaos in planar piecewise smooth vector fields

T. de Carvalho^a and R.D. Euzébio^b

^aDepartamento de Computação e Matemática, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, USP, Av. Bandeirantes, Zip Code 14098-322, Ribeirão Preto, SP, Brazil;

^bDepartamento de Matemática, IME–UFG, R. Jacarandá, Campus Samambaia, Zip Code 74001-970, Goiânia, GO, Brazil

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ABSTRACT

Some aspects concerning chaos and minimal sets in discontinuous dynamical systems are addressed. The orientability dependence of trajectories sliding through some variety is exploited and new phenomena emerging from this situation are highlighted. In particular, although chaotic flows and nontrivial minimal sets are not allowed in the plane, the existence of such objects for some classes of vector fields is verified. A characterization of chaotic flows in terms of orientable minimal sets is also provided. The main feature of the dynamical systems under study is related to the non uniqueness of trajectories in some zero measure region as well as the orientation of orbits reaching such region.

KEYWORDS

vector fields; piecewise smooth vector fields; chaos; minimal sets

1. Introduction

Dynamical systems have become one of the most promising areas of mathematics since its strong development started by Poincaré (see [23]). The main reason for this is due to the fact that several applied sciences from economy and biology to engineering and statistical mechanics benefited of dynamical systems' tools. In the last case, for instance, ergodic theory plays an important role, we mention for short Poincaré recurrence theorem as well as the concepts of chaos and entropy. In fact, while a mathematical object models a concrete phenomena, such modeling is in fact no more than an theoretical approximation of an real event and invariably ignores some important features of it. Is therefore mandatory to search for news methods and tools that are not only more realistic but also feasible in theory.

In this direction have emerged within the theory of dynamical system a set of methods which is now widely known by *piecewise smooth vector fields* (PSVFs, for short). For a formal introduction to PSVFs see [14]. The main advantage of PSVFs over the classical theory of dynamical system is the fact that they provide a more accurate approach by allowing non smoothness or discontinuities of the vector field defining the system. Indeed, several problems involving impact, friction or abrupt changes of certain regime can be modeled or at least approximated by PSVFs, in the

sense that the transition from one kind of behavior to another one can be idealized as a discrete and instantaneous transition. A non exhaustive list of applications of such theory involves the relay systems, the control theory, the stick-slip process, the dynamics of a bouncing ball and the antilock braking system (ABS), see those and other applications in [2], [3], [4], [11], [12], [15], [16], [18], [19] and references therein.

The main aspect of PSVFs concerns non uniqueness of solutions on some zero measure variety and consequent amalgamation of orbits under such region, which split the phase portrait into two or more pieces. That leads to the behavior known as *sliding motion*, characterized by the collapse of distinct trajectories which combine to slide on the common frontier of each dynamic. Under this scenario some behavior strange to the classical theory of dynamical systems may occur, so the study of new objects and the validation of known results is mandatory when one investigate PSVFs. For instance, we mention the Peixoto's Theorem (see [22]), the Closing Lemma (see [8]) and the Poincaré-Bendixson Theorem (see [5]), which posses analogues version in the context of PSVFs (see also [10,13,17]). We also mention that the study of PSVFs may take into account orientability of trajectories. This is because the collision of any particular trajectory to the boundary region and subsequent sliding occurs in different ways when considering forward or backward time.

This paper is addressed to some particular features of PSVFs. Indeed, we take into account aspects of chaotic PSVFs and how this concept relates to minimal sets. To do this, the definitions of both chaos e minimal sets are refined to consider the role of orientation and we provide a definitive characterization of chaotic PSVFs involving such objects.

Let $med(W)$ be the Lebesgue measure of a set W . The first main result of the paper states that *a PSVF Z is chaotic on the set W if, and only if, Z is positive chaotic and negative chaotic on W* . The second main result of the paper states that *if Z is chaotic on the set W and $med(W) > 0$ then W is positive minimal and negative minimal*. In order to prove these results we present and prove some other results which are indispensable to main results but also important on their own. For instance, we provide a sufficient condition for a Lebesgue measure subset of \mathbb{R}^2 to be chaotic, which elucidates the richness of PSVFs. Other considerations and results are presented timely throughout the text.

The paper is organized as follow: In Section 2 we provide the first statements around the subject of PSVFs, particularly considering minimal set for PSVFs and their chaotic behavior. In Section 3 we state and prove the main results of the paper and some consequences of them. In Section 4 we provide a discussion around the results of the paper and present some examples and counterexamples contextualizing the results.

2. Preliminaries

2.1. Piecewise smooth vector fields

Consider two smooth vector fields X and Y and a codimension one manifold $\Sigma \subset \mathbb{R}^2$ that separates the plane in two regions Σ^+ and Σ^- . A PSVF Z is a vector field defined in \mathbb{R}^2 and given by

$$Z(x, y) = \begin{cases} X(x, y), & \text{for } (x, y) \in \Sigma^+, \\ Y(x, y), & \text{for } (x, y) \in \Sigma^-. \end{cases} \quad (1)$$

In fact, since Σ is a codimension one manifold, there exists a function f such that $\Sigma = f^{-1}(0)$ and 0 is a regular value of f . As consequence, $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ . The basic results of differential equations in this context were stated by Filippov in [14], that we summarize next. Indeed, consider the Lie derivatives $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and $X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle, i \geq 2$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^2 . We distinguish the following regions on the discontinuity set Σ :

- (i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .
- (ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e .
- (iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s .

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s . It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$.

We say that $q \in \Sigma$ is a Σ -*regular point* if it is a sewing point or a regular point of the Filippov vector field. Lastly, any point $q \in \Sigma^p$ is called a *pseudo-equilibrium* of Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$. If there exist an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching q in a finite time, then such tangency is called a *visible tangency* for X (respec. Y); otherwise we call q an *invisible tangency* for X (respec. Y).

Definition 2.1. The **local trajectory** $\phi_Z(t, p)$ of a PSVF given by (1) is defined as a concatenation of orbits of X , Y and Z^Σ . A preserving-orientation union of local trajectories $\sigma_i(t, p_i)$ of the form $\Gamma_Z(t, p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t, p_i); t_i \leq t \leq t_{i+1}\}$ is called a **global trajectory** passing through p_0 , where $\sigma_i(t, p_i)$ satisfies $\sigma_i(t_{i+1}, p_i) = \sigma_{i+1}(t_{i+1}, p_{i+1}) = p_{i+1}$ and $t_i \rightarrow \pm\infty$ as $i \rightarrow \pm\infty$. A global trajectory Γ_Z^+ is a **positive** (respectively, Γ_Z^- **negative**) **global trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.

2.2. Minimal sets and chaotic PSVFs

One of the most important facts concerning PSVFs is the orientation of its trajectories. Indeed, it is very important, for instance, for the concept of invariance or defining the flow associated to the Filippov vector field. In the smooth theory of vector fields this distinction does not play an important role since we have uniqueness of trajectories. In this direction, we should verify if such distinction is also necessary when defining minimal sets and chaotic PSVFs. Indeed, these concepts do not play the same role by considering positive and negative times. As far as the authors know, the role of orientability under this context have not be treated in literature about PSVFs, although the concept of chaos and minimality have been discussed before, for instance, in [5], [6] and [10]. We start doing some adaptations to the definitions of invariance and minimality.

Definition 2.2. A set $A \subset \mathbb{R}^2$ is **positive invariant** (respectively, **negative invariant**) if for each $p \in A$ and all positive global trajectory $\Gamma_Z^+(t, p)$ (respectively, negative global trajectory $\Gamma_Z^-(t, p)$) passing through p it holds $\Gamma_Z^\pm(t, p) \subset A$ (respec-

tively, $\Gamma_Z^-(t, p) \subset A$). A set $A \subset \mathbb{R}^2$ is **invariant** for Z if it is positive and negative invariant.

Definition 2.3. Consider $Z \in \Omega$. A non-empty set $M \subset \mathbb{R}^2$ is **minimal** (respectively, either **positive minimal** or **negative minimal**) for Z if it is compact, invariant (respectively, either positive invariant or negative invariant) for Z and does not contain proper compact invariant subsets.

Next we present the definitions around chaotic PSVFs. As commented before, we need to distinguish between forward and backward time or assuming both possibilities. The notion of chaos we take into account is that based on Devaney. So, the first aspect to be considered is related to topological transitivity.

Definition 2.4. System (1) is **topologically transitive** on an invariant set W if for every pair of nonempty, open sets U and V in W , there exist $q^+, q^- \in U$, $\Gamma_Z^+(\cdot, q^+), \Gamma_Z^-(\cdot, q^-)$ global trajectories and $t_0^+ > 0 > t_0^-$ such that $\Gamma_Z^+(t_0^+, q^+)$ and $\Gamma_Z^-(t_0^-, q^-) \in V$.

Definition 2.5. System (1) is **topologically positive transitive** (respectively, **topologically negative transitive**) on a positive invariant (respectively, negative invariant) set W if for every pair of nonempty, open sets U and V in W , there exist $q \in U$, $\Gamma_Z^+(t, q)$ a positive (respectively, $\Gamma_Z^-(t, q)$ a negative) global trajectory and $t_0 > 0$ (resp., $t_0 < 0$) such that $\Gamma_Z^+(t_0, q) \in V$ (resp., $\Gamma_Z^-(t_0, q) \in V$).

Analogously to the definition of topologically transitive systems, the definition of sensitive dependence for PSVFs is inspired in the classical Devaney concept of chaos.

Definition 2.6. System (1) exhibits **sensitive dependence** on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < \text{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist $y^+, y^- \in B_\varepsilon(x) \cap W$ and global trajectories $\Gamma_x^+, \Gamma_x^-, \Gamma_{y^+}^+$ and $\Gamma_{y^-}^-$ passing through x, y^+ and y^- , respectively, satisfying

$$d_H(\Gamma_x^+(t), \Gamma_{y^+}^+(t)) = \sup_{a \in \Gamma_x^+(t), b \in \Gamma_{y^+}^+(t)} d(a, b) > r,$$

$$d_H(\Gamma_x^-(t), \Gamma_{y^-}^-(t)) = \sup_{a \in \Gamma_x^-(t), b \in \Gamma_{y^-}^-(t)} d(a, b) > r,$$

where $\text{diam}(W)$ is the diameter of W and d is the Euclidean distance.

Associated to the previous definition we give the next one, where the orientation of the trajectories of Z is also considered:

Definition 2.7. System (1) exhibits **sensitive positive dependence** (resp., **sensitive negative dependence**) on a compact positive invariant (resp., negative invariant) set W if there is a fixed $r > 0$ satisfying $r < \text{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive (resp., negative) global trajectories Γ_x^+ and Γ_y^+ (resp., Γ_x^- and Γ_y^-) passing through x and y , respectively, satisfying

$$d_H(\Gamma_x^+(t), \Gamma_y^+(t)) = \sup_{a \in \Gamma_x^+(t), b \in \Gamma_y^+(t)} d(a, b) > r$$

$$(\text{ resp., } d_H(\Gamma_x^-(t), \Gamma_y^-(t)) = \sup_{a \in \Gamma_x^-(t), b \in \Gamma_y^-(t)} d(a, b) > r),$$

where $diam(W)$ is the diameter of W and d is the Euclidean distance.

In this paper we will consider the notations stated in the following table.

Table of Abbreviations	
Topologically transitive	TT
Topologically positive transitive	TPT
Topologically negative transitive	TNT
Sensitive dependence	SD
Sensitive positive dependence	SPD
Sensitive negative dependence	SND

We should mention, as observed in [10], that Definitions 2.4 and 2.6 coincide with the definitions of topological transitivity and sensible dependence of smooth vector fields for single-valued flows, so these definitions are natural extension for a set-valued flow. Lastly, in what follows we introduce the definition of chaos and orientable chaos in the piecewise smooth context:

Definition 2.8. System (1) is **chaotic** (resp., either **positive chaotic** or **negative chaotic**) on a compact invariant (resp., either positive invariant or negative invariant) set W if it is TT and exhibits SD (resp., either TPT and exhibits SPD or TNT and exhibits SND) on W .

3. Main Results

In this Section we present and prove the main results of the paper.

Proposition 3.1. *Let \mathcal{A} be the set of pseudo cycles Γ of $Z = (X, Y)$ such that $\Gamma \cap (\overline{\Sigma^e} \cup \overline{\Sigma^s}) = \emptyset$ and Γ has a visible two-fold singularity. The elements Γ of \mathcal{A} are chaotic for Z .*

Proof. Let A, B open sets relative to Γ . Since Γ is a pseudo-cycle, given points $p_A \in A$ and $p_B \in B$, there exists a trajectory of Z connecting them (for positive and negative times). So Γ is topologically transitive.

On the other hand, given $x, y \in A$, there exists a trajectory passing through x and another trajectory passing through y such that each one of them follows a distinct path after the visible two-fold singularity of Γ . So Γ has sensitive dependence.

Therefore, Γ is chaotic. □

Remark 1. By the previous proposition, we conclude the existence of trivial minimal sets presenting chaotic behavior.

Proposition 3.2. *Let Z be a PSVF. The following statements hold*

- (a) Z is TT on W if, and only if, Z is simultaneously TPT and TNT on W ;
- (b) Z exhibits SD on W if, and only if, Z exhibits simultaneously SPD and SND on W ;

An analogous of Proposition 3.2 does not hold for minimal sets. Indeed, while sets which are both positive and negative minimal are also minimal, the converse is not true. Again, Example 2 of [6] exemplify this situation. Following we prove Proposition 3.2.

Proof of Proposition 3.2. We start proving item (a). First, assume that Z is TT on W and consider disjoint open sets U and V contained in W . Consequently, there exist points p_+ and p_- in U , times $t_+ > 0$ and $t_- > 0$ and trajectories Γ^+ and Γ^- satisfying $\Gamma^+(t_+, p_+) \in V$ and $\Gamma^-(-t_-, p_-) \in V$, where Γ^+ and Γ^- are positive and negative trajectories passing, respectively, through the points p_+ and p_- . Moreover, according to Remark 1 of [6], W is simultaneously positive and negative invariant. Consequently U and V can be connected through the positive trajectory Γ^+ starting on p_+ after a time t_+ , that is, Z is TPT on W . Analogously Z is TNT on W .

The converse is straightforward according to Definitions 2.4 e 2.5.

The proof of the statement (b) follows exactly the same ideas exposed at the proof of item (a). \square

Proposition 3.2 allows us to prove the following result.

Theorem 3.3. *A PSVF Z is chaotic on W if, and only if, Z is positive chaotic and negative chaotic on W .*

Proof of Theorem 3.3. First, assume that Z is chaotic on W . By Definition 2.8, Z is TT on W and exhibits SD on W . Thus, by Proposition 3.2, Z is simultaneously TPT and TNT on W and exhibits both SPD and SND on W . Then, using again Definition 2.8, we obtain that Z is positive chaotic and negative chaotic on W . The converse is straightforward. \square

The most part of the results obtained in [5] and [6] takes into account sets having positive Lebesgue measure. Indeed, in almost every approach concerning ergodic aspects of PSVFs, this is the interesting case. We cite, for instance, the existence of non-trivial minimal sets and planar chaotic PSVFs, as shown in the papers cited previously. In this direction we state the next result.

Lemma 3.4. *Let $K \subset \mathbb{R}^2$ be a compact invariant set and Z a PSVF. If $\text{med}(K) = 0$ and $K \notin \mathcal{A}$ then Z is not chaotic on K .*

We recall that \mathcal{A} is the set of pseudo-cycles having a visible two fold singularity which does not connect to any sliding or escaping segment (see Proposition 3.1).

Proof. First, suppose that $K \cap \Sigma \subset \Sigma^c \cup \Sigma^t$ and take $p \in K$. Consequently, the flow associated to Z , namely $\phi_t^Z(p)$ with $\phi_0^Z(p) = p$, satisfies $\phi_t^Z(p) \xrightarrow{t \rightarrow \infty} \Omega \in \omega(p) \subset K$, since K is compact. Here $\omega(p)$ denotes the ω -limit set of the point p . Thus, by using Poincaré Bendixson Theorem for PSVFs (see [5]) we get that Ω is a (pseudo-)equilibrium, a (pseudo-)graph or (pseudo-)cycle which does not belongs to \mathcal{A} since $\Omega \subset K$ and $K \notin \mathcal{A}$ by hypothesis. In any case, it is trivial to see that Z is not chaotic on K since Z does not exhibits SD on K .

Now consider the case where $K \cap (\Sigma^s \cup \Sigma^e) \neq \emptyset$ and suppose that there exist a PSVF Z which is chaotic on K . Take $p \in K \cap (\Sigma^s \cup \Sigma^e)$ and $V_p \subset \mathbb{R}^2$ a neighborhood of p . We will show that K is not invariant, more specifically, that there exist a trajectory $\tilde{\phi}_t$ passing through p and $t^* \in \mathbb{R}$ such that $\phi_{t^*}(\tilde{p}) = p$ with $\tilde{p} \notin K$. Indeed, consider the sets $V_p^+ = \{\phi_t^+(p) \cap V \mid \phi_t^+$ is the positive trajectory of Z passing

through q and V_p^- defined analogously for the negative trajectory. Observe that $\text{med}(V_p^+ \cup V_p^-) > 0$, since using the Definition 2.1, in this case the preimage of $K \cap (\Sigma^s \cup \Sigma^e)$ contain an open set $U \subset V_p$ satisfying $0 < \text{med}(U) < \text{med}(V_p^+ \cup V_p^-)$. Consequently there exist a point $q \in V_p^+ \cup V_p^-$ such that $q \notin K$, because otherwise $V_p^+ \cup V_p^- \subset K$ and then $\text{med}(K) > \text{med}(V_p^+ \cup V_p^-) > 0$ (see Figure 1). \square

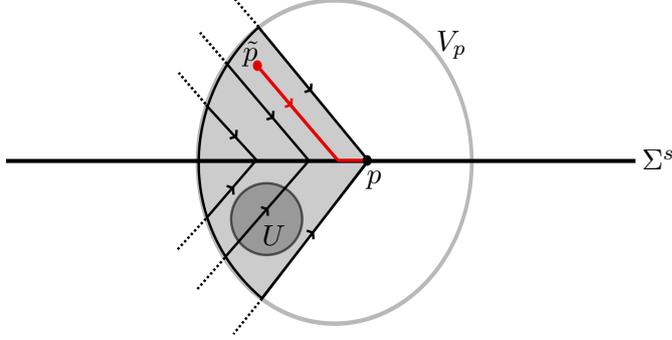


Figure 1. The neighborhood V_p of p . The filled region correspond to V_p^- , and in this case $V_p^+ = V_p \cap \Sigma$. Observe that it has positive Lebesgue measure. The trajectory in red correspond to $\tilde{\phi}_t$.

We observe that Theorem 3.4 does not make sense if Z is smooth, once there is no bi-dimensional smooth chaotic flow.

Before announce Theorem 3.6 let us prove the following lemma that will be useful in its proof, besides being elegant itself.

Lemma 3.5. *Let Z a chaotic PSVF on W . Then Z is chaotic on every compact invariant proper subset $\tilde{W} \subset W$.*

Proof. Suppose that Z is not chaotic on \tilde{W} . So, by Definition 2.8 we get that Z is not TT or does not presents SD on \tilde{W} . In any case, since $\tilde{W} \subset W$ we also get that Z is not TT or does not presents SD on W . This is a contradiction with the hypothesis. \square

In [6], among other results, the authors prove that, if a compact invariant set W satisfying $\text{med}(W) > 0$ is simultaneously positive and negative minimal for a PSVF Z , then Z is chaotic on W . Now, we prove the converse of this important theorem, as says Theorem 3.6 in what follows. Observe that, due to Theorem 3.4, we must impose a condition demanding the positive Lebesgue measure of the considered set.

Theorem 3.6. *If Z is chaotic on the compact invariant set W and $\text{med}(W) > 0$, then W is positive minimal and negative minimal for Z .*

Proof. According to Theorem 3.3, Z is positive chaotic on W . So, W is compact, non-empty and positive invariant. Suppose that W is not positive minimal. In this case, there exists a proper subset \tilde{W} of W with the previous three properties. Moreover, by Lemma 3.5 and Theorem 3.4, we get $\text{med}(\tilde{W}) > 0$ or $\text{med}(\tilde{W}) = 0$ and $\tilde{W} \in \mathcal{A}$. Of course \tilde{W} is not dense in W since \tilde{W} is compact and $\tilde{W} \neq W$. Therefore there exists an open set $A \subset W$ such that $A \cap \tilde{W} = \emptyset$. First suppose that $\text{med}(\tilde{W}) > 0$ and let $B \subset \tilde{W}$ be an open set of W . In this case, using the open sets A and B , we have that Z is not TPT. But this is a contradiction with the fact that Z is chaotic on W . On

the other hand, if $\text{med}(\widetilde{W}) = 0$, then $\widetilde{W} \in \mathcal{A}$ and consequently its interior $\text{Int}(\widetilde{W})$ is an open set satisfying $\text{med}(\text{Int}(\widetilde{W})) > 0$. So we can take the open sets $B \subset \text{Int}(\widetilde{W})$ and A as before to lead again to a contradiction with the fact that Z is chaotic on W . Therefore, W is positive minimal for Z .

An analogous argument proves that W is negative minimal for Z . \square

Next corollary is a straightforward consequence of Theorem 3.6, but it is very important once it provides a ultimate answer about the relation between chaotic systems and minimal sets.

Corollary 3.7. *If Z is chaotic on W and $\text{med}(W) > 0$ then W is minimal for Z .*

Proof. It is enough to use Theorem 3.6 and Lemma 2 of [6]. \square

We remark that the converse is not true, as observed in [6].

Next two corollaries are also consequences of Theorem 3.6. Their proof, analogously, are quite trivial although the results can find applications.

Corollary 3.8. *If $\text{med}(W) > 0$ and Z has a pseudo equilibria on W , then Z is not chaotic on W .*

Proof. It is not difficult to see that a pseudo equilibria is neither positive nor negative minimal for Z . Therefore the proof follows straightforward from Theorem 3.6. \square

Corollary 3.9. *If Z is positive (resp. negative) chaotic on W and $\text{med}(W) > 0$, then W is positive (resp. negative) minimal.*

Proof. It is enough repeat the proof of Theorem 3.6. \square

The next result provide a sufficient condition in order to a PSVF Z be chaotic on an invariant compact set W . Additionally, it guarantee that under suitable hypotheses the periodic trajectories of Z are dense in W .

Theorem 3.10. *Let Z be a PSVF and W a compact positive (respect. negative) invariant set. Given $x, y \in W$, assume that there exist a positive (respect. negative) trajectory ϕ_t^+ (respect. ϕ_t^-) connecting x and y . Then Z is positive (respect. negative) chaotic on W and the positive (respect. negative) periodic trajectories of Z are dense in W .*

The last theorem is inspired in Lemma 1 and Theorems 8 and 10 of [6] and its proof is analogous to the proofs of these results, so we will not prove it here. Also, Theorem 3.10 leads to the next corollary:

Corollary 3.11. *Let Z be a PSVF and W a compact invariant set on which any two points can be connected simultaneously by positive and negative trajectories. Then Z is chaotic on W and its periodic trajectories are dense in W .*

Proof. Since every pair of points in W can be connected simultaneously by positive and negative trajectories of Z , by Theorem 3.10, the PSVF Z is both positive and negative chaotic on Z . So, by Theorem 3.3, we get that Z is chaotic on W . Moreover, since the positive and negative periodic trajectories of Z are dense in W , the density of the periodic trajectories of Z on W is straightforward. \square

4. Discussions

We observed throughout the paper a closed relation between PSVFs presenting minimal sets or chaotic behavior. However, in order to observe the richness of such relation we introduced finer concepts by considering the orientation of the trajectories in time. By one hand, according to Theorem 14 of [6], every PSVF having a positive and negative non trivial minimal set K is chaotic on K . On the other hand, in this paper, due to Theorems 3.3 and 3.6 we get the equivalence. Putting those and other results of this paper together, we get the following diagram:

$$\begin{array}{ccccccc} Z \text{ is pos. and} & & Z \text{ is chaotic} & & W \text{ is pos. and} & & W \text{ is min.} \\ \text{neg. chaotic on } W & \Leftrightarrow & \text{on } W & \Leftrightarrow & \text{neg. min. for } Z & \Rightarrow & \text{for } Z \end{array}$$

We note by observing the previous diagram that it could exist some minimal set which is not chaotic for the PSVF, as the authors observed in [5]. Other aspects of that diagram are presented in what follows:

Orientable chaotic sets which are not chaotic: Consider the PSVF:

$$\begin{aligned} Z_\epsilon(x, y) = (\dot{x}, \dot{y}) = \frac{1}{2} & \left((-1, -2x - x^2(4x + 3) + (1 + \epsilon)x(3x + 2)) \right. \\ & \left. + \text{sgn}(y) (3, -2x + x^2(4x + 3) - (1 + \epsilon)x(3x + 2)) \right) \end{aligned} \quad (2)$$

or, equivalently,

$$Z_\epsilon(x, y) = \begin{cases} X(x, y) = (1, -2x) & \text{if } y \geq 0, \\ Y_\epsilon(x, y) = (-2, -x^2(4x + 3) + (1 + \epsilon)x(3x + 2)) & \text{if } y \leq 0, \end{cases} \quad (3)$$

with $\epsilon \in \mathbb{R}$ an arbitrarily small parameter. In [6] the authors proved that Z_0 has a chaotic set given (see Figure 2) by

$$\Lambda = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \text{ and } x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \quad (4)$$

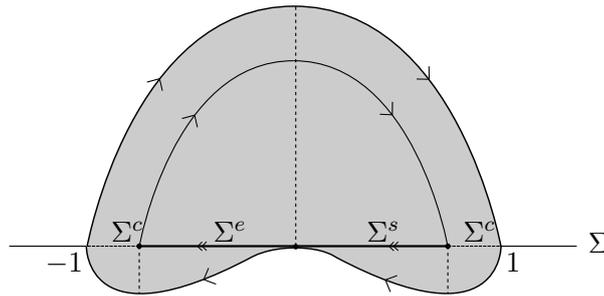


Figure 2. Chaotic set Λ .

Taking $\epsilon < 0$ (resp., $\epsilon > 0$) in (3) the PSVF Z_ϵ has a negative chaotic (resp., positive chaotic) set $\tilde{\Lambda}$, see the shadowed region in Figure 3 (resp., Figure 4), bounded by $\widehat{p_1 p_2} \cup \widehat{p_2 p_3} \cup \widehat{p_3 p_4} \cup \widehat{p_4 p_1}$, where $\widehat{a b}$ is the orbit-arc connecting the points a and

b. Despite of this, when $\epsilon \neq 0$, $\tilde{\Lambda}$ is not a chaotic set. This happens because $\tilde{\Lambda}$ is not an invariant set; it is only negative invariant (resp., positive invariant).

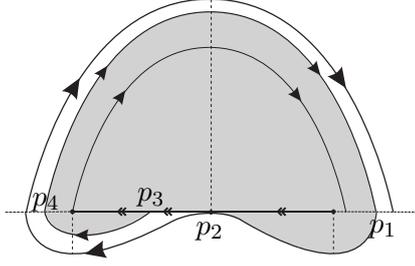


Figure 3. Negative chaotic set $\tilde{\Lambda}$.

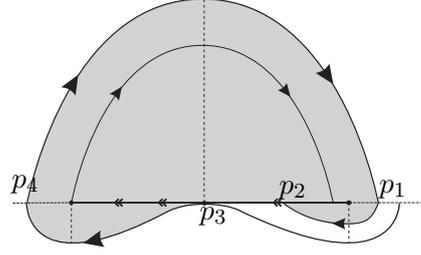


Figure 4. Positive chaotic set $\tilde{\Lambda}$.

Remark 2. The previous paragraph remains true if we change the word chaotic by the word minimal. A complete bifurcation analysis of the family (3) is given in [8].

The sets given in Figures 3 and 4 are orientable chaotic and orientable minimal sets. Despite of this, it is easy to exhibit examples of orientable minimal sets that are not orientable chaotic.

Orientable chaotic sets and orientable minimality: Consider the PSVF

$$Z(x, y) = (X(x, y), Y(x, y)) = ((-1, 3x^2 - 3), (1, -(9/4) + 3(-1 + x)x)).$$

Such PSVF has a periodic orbit (see Figure 5) which is a negative minimal set. However, Z is not a negative chaotic PSVF on the periodic orbit since it does not present SPD.

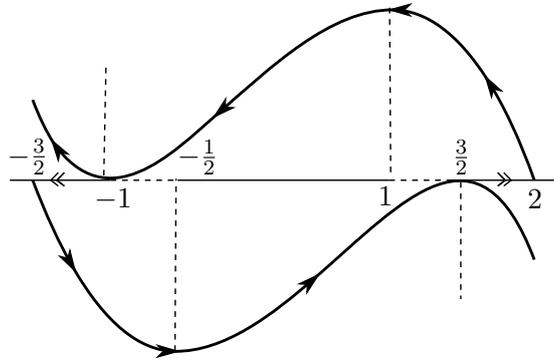


Figure 5. Periodic orbit (for positive time)

Observe that, in the last example the Lebesgue measure of the periodic orbit is null. However, it is not difficult to exhibit a minimal set W for some PSVF, with $med(W) > 0$, in such way that W is neither positive chaotic nor negative chaotic. Indeed, Example 2 of [6] satisfies these properties. In other words, in general minimality does not imply chaoticity. The converse, on the other hand, is true, as proved in Section 3.

Trivial chaos: In PSVFs the route to chaos is not hard. In fact, here we show that a chaotic behavior can be achieved by trivial minimal sets.

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