ON POINCARÉ-BENDIXSON THEOREM AND NON-TRIVIAL MINIMAL SETS IN PLANAR NONSMOOTH VECTOR FIELDS

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ABSTRACT. In this paper some qualitative and geometric aspects of non-smooth vector fields theory are discussed. A Poincaré-Bendixson Theorem for a class of nonsmooth systems is presented. In addition, a minimal set in planar Filippov systems not predicted in classical Poincaré-Bendixson theory and whose interior is non-empty is exhibited. The concepts of limit sets, recurrence and minimal sets for nonsmooth systems are defined and compared with the classical ones. Moreover some differences between them are pointed out.

1. INTRODUCTION

1.1. **Setting the problem.** Nonsmooth vector fields (NSVFs, for short) have become certainly one of the common frontiers between Mathematics and Physics or Engineering. Many authors have contributed to the study of NSVFs (see for instance the pioneering work [7] or the didactic works [1, 15], and references therein about details of these multi-valued vector fields). In our approach Filippov's convention is considered. So, the vector field of the model is discontinuous across a *switching manifold* and it is possible for its trajectories to be confined onto the switching manifold itself. The occurrence of such behavior, known as sliding motion, has been reported in a wide range of applications. We can find important examples in electrical circuits having switches, in mechanical devices in which components collide into each other, in problems with friction, sliding or squealing, among others.

For planar smooth vector fields there is a very developed theory nowadays. This theory is based on some important results. A now exhaustive list of such results include: *The Existence and Uniqueness Theorem, Hartman-Grobman Theorem, Poincaré-Bendixson Theorem* and *The Peixoto Theorem* among others (see the book [12]). A very interesting and useful subject is to answer if these results are true or not at the NSVFs scenario. It is already known that the statement of the first theorem is not true (see Example 1 and Figure 4 below) and the statement of the last theorem is true (under suitable

conditions, see [14]). Another extension to NSVFs of classical results on planar smooth vector fields include the concept of *Poincaré Index* of a vector field in relation to a curve, as stated in [3].

The specific topic addressed in this paper concerns with a Poincaré-Bendixson Theorem for NSVFs and non-trivial minimal sets which arise when the hypotheses of that theorem are not fulfilled. In smooth vector fields, under relatively weak hypothesis, Poincaré-Bendixson Theorem tells us which kind of limit set can arise on an open region of the Euclidean space $\mathbb{R}^2$. In particular, minimal sets in smooth vector fields are contained in the limit sets (as we show in this paper, it does not holds for NSVFs). In summary, by requiring some hypothesis concerning the switching manifold and avoiding sliding motion in some sense, we extend the classical Poincaré-Bendixson Theorem for a infinitely greater class of systems. In addition we show that, by allowing sliding motion, one must add up extra hypothesis in order to obtain a version of such theorem. Indeed, in the presence of sliding motion, we can exhibit NSVFs possessing positive Lebesgue measure minimal sets (which we call here *non-trivial*) whose trajectories confined on it present strange behavior, in the sense that it is not predicted in the classical theory of planar differential systems. Indeed, some of them are pointed out and compared with the classical theory of smooth vector fields. Lastly, we should mention that, up to the best of our knowledge, such special sets have not been considered in the literature until now.

The paper is organized as follows: In Subsection 1.2 the main results are stated. In Section 2 some of the standard theory on NSVFs, a brief introduction about Filippov systems and new definitions on this scenario are presented. Section 3 is devoted to prove Poincaré-Bendixson Theorem for NSVFs and discuss some aspects on it. In Section 4 an example of a NSVF presenting a non-trivial minimal set is exhibited and discussed.

1.2. **Statement of the main results.** In this paper we are concerned with limit sets and minimal sets of NSVFs on the plane. For the classical theory it is well known the Poincaré-Bendixson Theorem which establishes that the limit sets of a smooth vector field is either an equilibrium point or a periodic orbit or a graph. In the main result of our paper we have an analogous result for NSVFs allowing sliding motion only in a particular way. In fact, in this case we add to the classical limit sets a pseudo-graph, a pseudo-cycle and an equilibrium point under the switching manifold (for details, see Section 2).

Let $V \subset \mathbb{R}^2$ an open set containing the origin. We consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. In this paper we assume that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$(1) \qquad Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept that Z is multi-valued at points of Σ. The basic results of NSVFs were stated by Filippov in [7].

In the sequel we state the main results of the paper. They deal with limit sets of trajectories and limit sets of points.

Theorem 1. *Let $Z = (X,Y) \in \Omega$. Assume that Z has a maximal trajectory $\Gamma_Z(t,p)$ whose positive trajectory $\Gamma_Z^+(t,p)$ is contained in a compact subset $K \subset V$ and Z does not have sliding motion in a Z-invariant neighborhood of K. Suppose also that X and Y have a finite number of critical points in K and a finite number of tangency points with Σ. Then, the ω-limit set $\omega(\Gamma_Z(t,p))$ of $\Gamma_Z(t,p)$ is one of the following objects:*

   (i) *an equilibrium of X or Y;*
  (ii) *a periodic orbit of X or Y;*
 (iii) *a graph of X or Y;*
 (iv) *a pseudo-cycle;*
  (v) *a pseudo-graph;*
 (vi) *a singular tangency.*

For a precise definition of pseudo-cycle, pseudo-graph and singular tangency, see Section 2.

As a consequence of Theorem 1, since the uniqueness of orbits and trajectories passing through a point is not achieved, we have the following corollary (see definitions of orbits and trajectories in Section 2):

Corollary 1. *Under the same hypothesis of Theorem 1 the ω-limit set $\omega(p)$ of a point $p \in V$ is one of the objects described in items (i), (ii), (iii), (iv), (v) and (vi) or a finite union of them.*

The same holds for the α-limit set, reversing time.

For the general case where sliding motion is allowed in the compact set K, we can not exhibit an analogous result. In fact, as shown in Example 3 and in Propositions 1 and 2, there exist *non-trivial minimal sets* (i.e., minimal sets distinct from an equilibrium point or of a closed trajectory) in this scenario.

Remark 1. *Besides the classical version of the Poincaré-Bendixson Theorem conceived for vector fields defined in two dimensional manifolds (see [12]), there are also a version of such theorem for hybrid systems (see [11] and [13]). In fact, a hybrid system is a kind of piecewise smooth system*

but we stress that in both references the authors assume the existence of re-sets which switch the flows into different domains, which does not happen in our version of the Poincaré-Bendixson Theorem for PSVFs. Also, the possibilities we can obtain as limit sets include a union of distinct objects (see Corollary 1), which does not happen in the theorem of [13]. Another issue we can point out which distinguishes our work in comparison to these ones done for hybrid systems is that while our Theorem 1 ask for only some few hypoteheses, the version presented in [11] assume several of them. Still, the hypotheses assumed in [11] are stronger than the ones presented in [13], since in the last reference they do not allow Zeno states, where a sewed focus is reached for a finite time.

2. Preliminaries

Consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle, i \geq 2$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .
(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.
(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s (see Figure 1). It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $-Z$ and then we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^Σ for both cases. In our pictures we represent the dynamics of Z^Σ by double arrows.
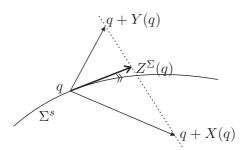

Figure 1. Filippov's convention.

A point $q \in \Sigma$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$ (q is a tangent contact point between the trajectories of X and/or Y with Σ).
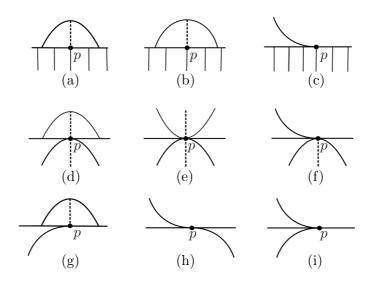
FIGURE 2. Cases where occur regular tangential singularities. The dashed lines represent the curves where $X.f(p) = 0$ or $Y.f(p) = 0$.
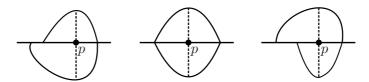

FIGURE 3. The particular cases where occur singular tangential singularities.

For a given $W \in \chi$, we say that a positive integer r is the *contact order* of the trajectory Γ_W of W with Σ at p if $W^k.f(p) = 0$, $\forall k = 0, \ldots, r-1$ and $W^r.f(p) \neq 0$. For $W = X$ (respec. Y) we say that $p \in \Sigma$ is *invisible tangency* if the contact order r of Γ_X (respec. Γ_Y) passing through p is even and $X^r.f(p) < 0$ (respec. $Y^r.f(p) > 0$). On the other hand, for $W = X$ (respec. Y) we say that $p \in \Sigma$ is *visible tangency* if the contact order r of Γ_X (respec. Γ_Y) passing through p is even and $X^r.f(p) > 0$ (respec. $Y^r.f(p) < 0$).

A tangential singularity $p \in \Sigma^t$ is *singular* if p is a invisible tangency for both X and Y. On the other hand, a tangential singularity $p \in \Sigma^t$ is *regular* if it is not singular. Except for the sake of orientation, Figures 2 and 3 illustrate all possible cases for regular and singular tangencies, respectively.

Remark 2. *Let p be in Σ and Γ_X (respec. Γ_Y) be the trajectory of X (respec. Y) passing through p. Consider $V_p = V_p^- \cup \{p\} \cup V_p^+$, where $V_p^- =$*

$\{x \in \Sigma; x < p\}$ and $V_p^+ = \{x \in \Sigma; x > p\}$. Let m be the sum of the contact order of the trajectories Γ_X of X and Γ_Y of Y with Σ at p. It is possible to give a characterization of the behavior of $Z \in \Omega$ in the neighborhood V_p of p in terms of m. In fact, if m is odd, then $V_p^- \subset \Sigma^c$ and $V_p^+ \subset \Sigma^s \cup \Sigma^e$ (or vice versa, depending on the orientation). See Figure 2, items (a), (b), (f) and (g). On the other hand, if m is even, we have three cases: (i) $V_p \setminus \{p\}$ is contained in Σ^c; (ii) $V_p \setminus \{p\}$ is contained either in Σ^s or Σ^e; (iii) $V_p^- \subset \Sigma^s$ and $V_p^+ \subset \Sigma^e$ or vice versa. This cases are represented in the Figures 3 and 2, items (c), (d), (e), (h) and (i).

If W is a vector field, then we denote its flow by $\phi_W(t, p)$. Thus,

$$\begin{cases} \dfrac{d}{dt}\phi_W(t, p) = W(\phi_W(t, p)), \\[2mm] \phi_W(t_0, p) = p, \end{cases}$$

where $t \in I = I(p, W) \subset \mathbb{R}$, the interval where the W-trajectory passing through $p \in V$ is defined.

Definition 2. *The **local trajectory (orbit)** $\phi_Z(t, p)$ of a NSVF given by (1) is defined as follows:*

- *For $p \in \Sigma^+ \setminus \Sigma$ or $p \in \Sigma^- \setminus \Sigma$ the trajectory is given by $\phi_Z(t, p) = \phi_X(t, p)$ or $\phi_Z(t, p) = \phi_Y(t, p)$ respectively, where $t \in I$.*
- *For $p \in \Sigma^c$ such that $X.f(p) > 0$, $Y.f(p) > 0$ and taking time $t = t_0$ at p, the trajectory is defined as $\phi_Z(t, p) = \phi_Y(t, p)$ for $t \in I \cap \{t \le t_0\}$ and $\phi_Z(t, p) = \phi_X(t, p)$ for $t \in I \cap \{t \ge t_0\}$. For the case $X.f(p) < 0$ and $Y.f(p) < 0$ the definition is the same reversing time.*
- *For $p \in \Sigma^e$ and taking time $t = t_0$ at p, the trajectory is defined as $\phi_Z(t, p) = \phi_{Z^\Sigma}(t, p)$ for $t \in I \cap \{t \le t_0\}$ and $\phi_Z(t, p)$ is either $\phi_X(t, p)$ or $\phi_Y(t, p)$ or $\phi_{Z^\Sigma}(t, p)$ for $t \in I \cap \{t \ge t_0\}$. For the case $p \in \Sigma^s$ the definition is the same reversing time.*
- *For p a regular tangency point and taking time $t = t_0$ at p, the trajectory is defined as $\phi_Z(t, p) = \phi_1(t, p)$ for $t \in I \cap \{t \le t_0\}$ and $\phi_Z(t, p) = \phi_2(t, p)$ for $t \in I \cap \{t \ge t_0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*
- *For p a singular tangency point $\phi_Z(t, p) = p$ for all $t \in \mathbb{R}$.*

Remark 3. *Note that for $p \in \Sigma^e \cup \Sigma^s$ and p a regular tangency, the trajectory Γ_Z passing through p can be chosen of many distinct ways.*

Definition 3. *A **global trajectory (orbit)** $\Gamma_Z(t, p_0)$ of $Z \in \Omega$ passing through p_0 is a union*

$$\Gamma_Z(t, p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t, p_i); t_i \le t \le t_{i+1}\}$$

of preserving-orientation local trajectories $\sigma_i(t, p_i)$ satisfying $\sigma_i(t_i, p_i) = p_i$ and $\sigma_i(t_{i+1}, p_i) = p_{i+1}$.

Definition 4. *A **maximal trajectory** $\Gamma_Z(t, p_0)$ is a global trajectory that cannot be extended to any others global trajectories by joining local ones, that is, if $\widetilde{\Gamma}_Z$ is a global trajectory containing Γ_Z then $\widetilde{\Gamma}_Z = \Gamma_Z$. In this case, we call $I = (\tau^-(p_0), \tau^+(p_0))$ the **maximal interval of the solution** Γ_Z, where*

$$\tau^{\pm}(p_0) = \lim_{i \to \pm\infty} t_i.$$

*A maximal trajectory is a **positive** (respectively, **negative**) **maximal trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$).*

One should note that the maximal interval of solution may not cover the interval $(-\infty, +\infty)$, that is, $\tau^{\pm}(p_0)$ could be finite values.

Definition 5. *Given $\Gamma_Z(t, p_0)$ a maximal trajectory, the set $\omega(\Gamma_Z(t, p_0)) = \{q \in V; \exists (t_n) \text{ satisfying } \Gamma_Z(t_n, p_0) \to q \text{ with } t_n \to \tau^+(p_0) \text{ when } n \to \infty\}$ (respectively $\alpha(\Gamma_Z(t, p_0)) = \{q \in V; \exists (t_n) \text{ satisfying } \Gamma_Z(t_n, p_0) \to q \text{ with } t_n \to \tau^-(p_0) \text{ when } n \to \infty\}$) is called ω-**limit** (respectively α-**limit**) **set of $\Gamma_Z(t, p_0)$**. The ω-limit (respectively α-limit) set **of a point p** is the union of the ω-limit (respectively α-limit) sets of all maximal trajectories passing through p.*

Definition 6. *Two PSVFs $Z = (X, Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \Omega$ defined in open sets $U, \widetilde{U}$ and with switching manifold Σ are Σ-**equivalent** if there exists an orientation preserving homeomorphism $h : U \to \widetilde{U}$ that sends $U \cap \Sigma$ to $\widetilde{U} \cap \Sigma$, the orbits of X restricted to $U \cap \Sigma^+$ to the orbits of $\widetilde{X}$ restricted to $\widetilde{U} \cap \Sigma^+$, and the orbits of Y restricted to $U \cap \Sigma^-$ to the orbits of $\widetilde{Y}$ restricted to $\widetilde{U} \cap \Sigma^-$.*

Remark 4. • *An important notice is that the sewed focus given in Figure 3 can be reached in a finite time, i.e., $\tau^{\pm}(p_0)$ is finite in the previous definitions. However, also there exist sewed focus that are reached for infinite time. Moreover, both these focus are Σ-equivalent. So, the limit set of both coincides. These kind of trajectories are called **Zeno state** in the literature.*
• *Also, neither a pseudo-graph nor a pseudo-cycle can be reached in a finite time. In fact, let $\Gamma = \Gamma(t, p)$ a pseudo-cycle (resp., pseudo-graph) containing a regular point p, with $t \in I_\Gamma$. Suppose that $p = \Gamma(0, p)$. So, there exists a time $\overline{t} > 0$ such that the arc of trajectory γ connecting p to $\overline{p} = \Gamma(\overline{t}, p)$ belongs to Γ. Given a neighborhood V_γ of γ, every trajectory converging to Γ spends a time $\tilde{t} \in (\overline{t} - \varepsilon, \overline{t} + \varepsilon)$ in V_γ, with ε a small positive number. Since every trajectory converging to Γ must return to V_γ infinitely many times, we get the result.*

Example 1. *Consider Figure 4. We observe that the maximal orbit passing through $q \in \Sigma$ is not necessarily unique. In fact, according to the third bullet of Definition 2, the positive local trajectory by the point $q \in \Sigma$ can provide three distinct paths, namely, Γ_1, Γ_2 and Γ_3. In particular, it is clear that the Existence and Uniqueness Theorem is not true in the scenario of NSVFs.*

Moreover, the ω-limit set of Γ_i, $i = 1, 2, 3$ is, respectively, a focus, a pseudo-equilibrium and a limit cycle and, consequently, the ω-limit set of q being the union of these objects is not a connected set. This fact is not predicted in the classical theory. Note that the α-limit set of q is a connected set composed by the pseudo-equilibrium p.
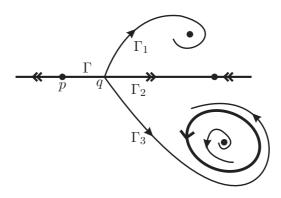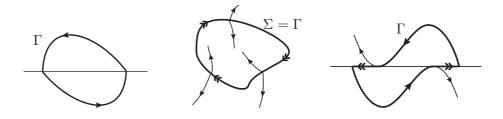

FIGURE 4. An orbit by a point is not necessarily unique.

Definition 7. *Consider $Z = (X, Y) \in \Omega$. A closed maximal orbit Δ of Z is a:*

(i) ***pseudo-cycle*** *if $\Delta \cap \Sigma \neq \emptyset$ and it does not contain neither equilibrium nor pseudo-equilibrium (See Figure 7).*

(ii) ***pseudo-graph*** *if $\Delta \cap \Sigma \neq \emptyset$ and it is a union of equilibria, pseudo equilibria and orbit-arcs of Z joining these points (See Figure 6).*

FIGURE 5. Possible kinds of pseudo-cycles.

Definition 8. *A set $A \subset \mathbb{R}^2$ is **Z-invariant** if for each $p \in A$ and all maximal trajectory $\Gamma_Z(t, p)$ passing through p it holds $\Gamma_Z(t, p) \subset A$.*

Definition 9. *A set $M \subset \mathbb{R}^2$ is **minimal for Z** $\in \Omega$ if*

(i) *$M \neq \emptyset$;*

(ii) *M is compact;*
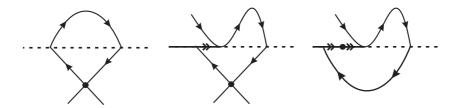
(iii) *M is Z-invariant;*

FIGURE 6. Examples of pseudo-graphs.

(iv) *M does not contain proper subset satisfying (i), (ii) and (iii).*

Remark 5. *Observe that the pseudo-cycle Γ on the center of Figure 5 is the α-limit set of all maximal trajectories on a neighborhood of it, however Γ is not Z-invariant according to Definition 8. This phenomenon point out a distinct and amazing aspect not predicted for the classical theory about smooth vector fields where the α and ω-limit sets are invariant sets.*

3. Considerations on the Poincaré-Bendixson Theorem for NSVFs

This section is dedicated to the Poincaré-Bendixson Theorem and present some considerations concerning the version of this important theorem in the NSVFs scenario, as well as its proof. In fact, we remark that Theorem 1 takes into account that the NSVF has no sliding motion on a neighborhood of the compact set K. Later on in this paper, we will see that some new and unpredictable phenomena in the classical theory of smooth vector fields could happen by considering sliding motion on the compact set K. Nevertheless, it happens because it is not possible to guarantee the uniqueness of trajectories in sliding points. That means that we can not generalize the Poincaré-Bendixson Theorem presented in Section 1.2 without assuming extra hypothesis.

As we have just commented, Theorem 1 holds if the compact set K does not contain (or connect to) sliding points, since otherwise it should lead to the existence of nonstandard phenomena. However, in the set K it is allowed to have trajectories which are tangent to the switching manifold, which sometimes leads to non-uniqueness of trajectories in such points; thus such theorem does not consider only trajectories crossing transversally the switching manifold but also reaching it tangentially. In fact, by considering only transversal arrivals, one can see that the trajectories may not return to such manifold, which means that they are not confined on a compact or accumulate in some object predicted by the classical version of the referred theorem, which are situations of minor interest in this paper.

The proof of Theorem 1 takes into account the classical Poincaré-Bendixson Theorem and the concept of Poincaré return map for NSVFs.

Proof of Theorem 1. Consider $p \in K \subset V$ and V_K the Z-invariant neighborhood of K stated in Theorem 1. If there exists a time $t_0 > 0$ such that the maximal trajectory $\Gamma_Z(t, p)$ by p does not collide with Σ for $t > t_0$ then we can apply the classical Poincaré-Bendixson Theorem in order to conclude that one of the three first cases (i), (ii) or (iii) happens. Otherwise, there exists a sequence $(t_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ of positive times, $t_i \to \tau^+(p_0)$, such that $p_i = \Gamma_Z(t_i, p) \in \Sigma$.

The hypothesis that Z does not have sliding motion on V_K implies $X.f(p_i) \cdot Y.f(p_i) \geq 0$ which separate the points $p_i = \Gamma_Z(t_i, p)$ in the following sets.

$$S_p \;=\; \{p_i; p_i \text{ is a singular tangency or an equilibrium of } X \text{ or } Y\},$$

$$T_p \;=\; \{p_i; \text{there is no ambiguity on the choice of the local}$$
$$\text{trajectory though } p_i \text{ and } p_i \notin S_p\},$$

$$N_p \;=\; \{p_i; p_i \notin S_p \cup T_p\}$$

Observe that, S_p has at most one element p_1 and, by hypothesis, N_p is a finite set. If $S_p \neq \emptyset$, then $p = \Gamma_Z(t, p) = p_1 = \omega\{p\}$. Otherwise, we separate the proof in two cases: T_p is finite and T_p is not finite. Assume that T_p is a finite set. We denote by n_p and t_p the number of elements of the sets N_p and T_p respectively. According to Definition 2, a maximal trajectory of Z by $p_l \in N_p$ can follow at most two distinct paths. Let us denote by Γ_m an arc of $\Gamma_Z(t, p)$ connecting two consecutive points p_i and p_{i+1}, $i \in \mathbb{N}$. In this case there exists at most $2^{n_p} + t_p$ arcs Γ_m of $\Gamma_Z(t, p)$. So, there exists a (sub)set $\Upsilon \subset \{1, 2, \ldots, 2^{n_p} + t_p\}$ such that $\Gamma = \bigcup_{j \in \Upsilon} \Gamma_j$ is a closed orbit intersecting Σ (i.e., a pseudo-cycle) contained in $\Gamma_Z(t, p)$ and with the property that $\Gamma_Z(t, p)$ visit each arc Γ_j of Γ an infinite number of times. In what follows we prove that $\omega(\Gamma_Z(t, p)) = \Gamma$. In fact, as $\Gamma_Z(t, p)$ must visit each arc Γ_j of Γ an infinite number of times then $\Gamma \subset \omega(\Gamma_Z(t, p))$.

On the other hand, if $x_0 \in \omega(\Gamma_Z(t, p))$ then there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to \tau^+(p_0)$, such that $\Gamma_Z(s_k, p) = x_k \to x_0$. Moreover, since $\Gamma_Z(t, p)$ also is composed by a finite number of arcs Γ_m, $s_k \to \tau^+(p_0)$ and $\Gamma_Z(t, p)$ has no equilibria (otherwise it does not visit Σ infinitely many times), there exists a subsequence (x_{k_j}) of (x_k) that visits some arcs Γ_m infinitely many times. Since Γ is a compact set, we get $x_0 \in \Gamma$.

Now assume that T_p is not a finite set. In this case, there exists a point $q \in \Sigma$ and a subsequence $(t_{i_j}) = (s_j)$ of (t_i) such that

$$(2) \qquad\qquad \lim_{j \to \infty} \Gamma_Z(s_j, p) = q$$

since $\Gamma_Z^+(t, p) \subset K$, a compact set. Observe that $q \in \omega(\Gamma_Z(t, p)) \cap \Sigma \neq \emptyset$. As Z does not have sliding motion on V_K, we get either $\{q\} = S_q$ or $q \in T_q$ or $q \in N_q$.

If $\{q\} = S_q$ is a singular tangency then both X and Y have an invisible tangency point at q. As consequence there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to \tau^+(p_0)$, such that $\Gamma_Z(s_k, p) \in \Sigma$ and $\Gamma_Z(s_k, p) = x_k \to q$, and then

there is a small neighborhood V_q of q such that all trajectory of Z that starts at a point of V_q converges to q. See Figure 3. Therefore, $\omega(\Gamma_Z(t, p)) = \{q\}$.
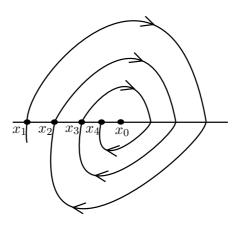

FIGURE 7. Case where there exists a singular tangency in $\omega(\Gamma_Z(t, p)) \cap \Sigma$.

In the sequel we separate the analysis in two cases: either $\Gamma_Z(t, q)$ contains equilibria or contains no equilibria. Consider the case when $\Gamma_Z(t, q)$ contains no equilibria.

If $q \in N_q$ then q is a visible tangency for both X and Y. So, there are two possible choices for the positive local trajectory of Z passing through q and at least one of them is such that it is contained in $\omega(\Gamma_Z(t, p))$. By continuity, the maximal trajectory $\Gamma_Z(t, q)$ that passes through q, contained in $\omega(\Gamma_Z(t, p))$, must come back to a neighborhood V_q of q in Σ. Moreover, by the Jordan Curve Theorem, $\Gamma_Z(t, q) \cap V_q = \{q\}$, otherwise there exists a flow box not containing q for which $\Gamma_Z(t, q)$ and, consequently, $\Gamma_Z(t, p)$, do not depart it. This is a contradiction with the fact that the orbit $\Gamma_Z(t, p)$ must visit any neighborhood of q infinite many times. Therefore, $\Gamma_Z(t, q)$ is closed (i.e., is a pseudo-cycle) and $\omega(\Gamma_Z(t, p)) = \Gamma_Z(t, q)$. If $q \in T_q$, then it is clear that the local trajectory passing through q is unique and again, by a similar argument, we can conclude that $\omega(\Gamma_Z(t, p)) = \Gamma_Z(t, q)$ is a pseudo-cycle.

The remaining case is when $\Gamma_Z(t, q)$ has equilibria either of X or Y. In this case for each regular point $\tilde{q} \in \omega(\Gamma_Z(t, p))$ consider the local orbit $\Gamma_Z(t, \tilde{q})$ which is contained in $\omega(\Gamma_Z(t, p))$. The set $\omega(\Gamma_Z(t, \tilde{q}))$ can not be a periodic orbit or a graph contained in Σ^+ or in Σ^-, because the orbit $\Gamma_Z(t, p)$ must visit any neighborhood of q infinite many times. So, the unique option is that $\omega(\Gamma_Z(t, \tilde{q})) = \{z_i\}$ where z_i is an equilibrium of X or of Y since otherwise, by the same arguments of the previous paragraph $\omega(\Gamma_Z(t, q))$ should be a closed orbit without equilibria which is a contradiction. Similarly, the α-limit set $\alpha(\Gamma_Z(t, \tilde{q})) = \{z_j\}$ where z_j is an equilibrium of X or of Y. Thus, with an appropriate ordering of the equilibria z_k, $k = 1, 2 \ldots, m$, (which

may not be distinct) and regular orbits $\Gamma_k \subset \omega(\Gamma_Z(t,p))$, $k = 1, 2 \ldots, m$, we have

$$\alpha(\Gamma_k) = z_k \quad \text{and} \quad \omega(\Gamma_k) = z_{k+1}$$

for $k = 1, \ldots, m$, where $z_{m+1} = z_1$. It follows that the maximal trajectory $\Gamma_Z(t,p)$ either spirals down to or out toward $\omega(\Gamma_Z(t,p))$ as $t \to \tau^+(p_0)$. It means that in this case $\omega(\Gamma_Z(t,p))$ is a pseudo-graph composed by the equilibria z_k and the arcs Γ_k connecting them, $k = 1, \ldots, m$.

This concludes the proof of Theorem 1. $\square$

Now we perform the proof of Corollary 1 (see Section 1.2). In Example 2 below we illustrate its consequences.

Proof of Corollary 1. In fact, since by Definition 5 the ω-limit set of a point is the union of the ω-limit set of all maximal trajectories passing through it, the conclusion is obvious. $\square$

Example 2. *Consider Figure 8. Here we observe a NSVF without sliding motion on Σ where the conclusions of Theorem 1 and Corollary 1 can be observed. Since the uniqueness of trajectories by p is not achieved (neither for positive nor for negative times) both the α and the ω-limit sets are disconnected sets. The α-limit set of p is composed by the focus α_1 and the singular tangency point α_2. The ω-limit set of p is composed by the saddle ω_1 and the periodic orbit Γ_1.*
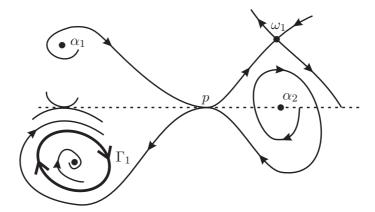

FIGURE 8. Both the α-limit set $\{\alpha_1, \alpha_2\}$ and the ω-limit set $\{\omega_1, \Gamma_1\}$ of the point p are disconnected. Sliding motion on Σ is not allowed.

When we allow sliding motion on K, each subset $N \subset \Sigma^e \cup \Sigma^s$ is necessarily not invariant, because on this region there is no uniqueness of solution. Actually, if we take a point $q \in N$, there will exist infinitely many solutions passing through q when the time goes to future or past (see, for instance, Examples 1 and Remark 5). For this reason, it is possible to occur some interesting phenomena where classical properties of both limit and minimal

sets do not work. In particular can be not possible to establishes a version of the Poincaré-Bendixson Theorem as stated in the classical theory for this scenario.

4. Minimal Sets with non-empty interior

Finding limit sets of trajectories of vector fields is an important task inside the qualitative theory of dynamical systems. In the literature there are several recent papers (see for instance [4, 5, 8, 10]) where the authors explicitly exhibit the phase portraits of some NSVFs with their unfoldings. However, all the limit sets exhibited have trivial minimal sets (i.e., the minimal sets are equilibria, pseudo equilibria, cycles or pseudo-cycles). At this section we present a non-trivial minimal set in the NSVFs scenario which has non-empty interior and does not look like neither a cycle nor an equilibrium.

We stress that the minimal set presented in this section works also as an example where we do not fulfill one of the hypotheses of the Poincaré-Bendixson Theorem for NSVFs. Indeed, in such example the compact set (which we will prove to be minimal) contain a segment of sliding points. Therefore, Theorem 1 can not be extended without assuming extra hypothesis.

Example 3. *Consider $Z = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 2x - 4x^3)$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. The parametric equation for the integral curves of X and Y with initial conditions $(x(0), y(0)) = (0, k_+)$ and $(x(0), y(0)) = (0, k_-)$, respectively, are known and its algebraic expressions are given by $y = -x^2 + k_+$ and $y = x^4/2 - x^2/2 + k_-$, respectively. It is easy to see that $p = (0, 0)$ is an invisible tangency point of X and a visible one of Y. It is also easy to note that the points $p_\pm = (\pm\sqrt{2}/2, 0)$ are both invisible tangency points of Y. Note that between p_- and p there exists an escaping region and between p and p_+ a sliding one. Further, every point between $(-1, 0)$ and p_- or between p_+ and $(1, 0)$ belong to a sewing region. Consider now the particular trajectories of X and Y for the cases when $k_+ = 1$ and $k_- = 0$, respectively. These particular curves delimit a bounded region of plane that we call Λ and it is the main object of this section. Figure 9 summarizes these facts.*

The previous example is also exhibited in [2] where we prove the existence of chaotic PSVFs in the plane.

Proposition 1. *Consider $Z = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 2x - 4x^3)$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. The set*

$$(3) \quad \Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \ \text{ and } \ x^4/2 - x^2/2 \leq y \leq 1 - x^2\}.$$

is a minimal set for Z.

Proof. It is easy to see that Λ is compact and has non-empty interior. Moreover, by Definition 2, on $\partial\Lambda \setminus \{p\}$ we have uniqueness of trajectory (here ∂B means the boundary of the set B). Note that a maximal trajectory of
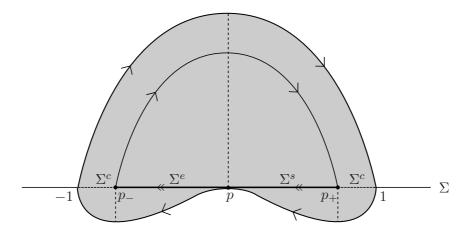
FIGURE 9. Special integral curves and tangency points.

any point in Λ meets p for some time t^*. Since p is a visible tangency point for Y and $p \in \overline{\partial\Sigma^e} \cap \overline{\partial\Sigma^s}$, according to the fourth bullet of Definition 2 any trajectory passing through p remain in Λ. Consequently Λ is Z-invariant. Moreover, given $p_1, p_2 \in \Lambda$ the positive maximal trajectory by p_1 reaches the sliding region between p and p_+ and slides to p. The negative maximal trajectory by p_2 reaches the escaping region between p and p_- and slides to p. So, there exists a maximal trajectory connecting p_1 and p_2. Now, let $\Lambda' \subset \Lambda$ be a Z-invariant set. Given $q_1 \in \Lambda'$ and $q_2 \in \Lambda$ since there exists a maximal trajectory connecting them we conclude that $q_2 \in \Lambda'$. Therefore, $\Lambda' = \Lambda$ and Λ is a minimal set. $\qquad\square$

Of course, the existence of such a set Λ is due to the non-uniqueness of trajectories in sliding points and at the first moment it has no other implications but only prints an idea of *non-determinism* of the trajectories on Σ^e and Σ^s. However, we highlight that, once the orbits delimit a compact set containing a sliding/escaping region, it leads us to a very rich dynamics which includes not only strange limit sets but also non-trivial recurrence and positive measure minimal sets, which have been not verified in the planar theory of dynamical systems. In some sense, the set Λ says also that Denjoy-Schwartz Theorem (see [6, 9]) can not be extended without assuming extra hypotheses on the smoothness of the vector fields under study, therefore such objects should be distinguished in the study of NSVFs.

Other exotic examples can be easily obtained. In Figure 10 we show another phase portrait of a NSVF presenting a non-trivial minimal set with non-empty interior.

Next result highlights one of the features of the set Λ.

Proposition 2. *Let Λ given by (3). If $q \in \Lambda$ then there exists a trajectory passing trough q that is not dense in Λ.*
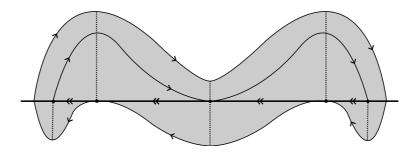
FIGURE 10. Non-trivial minimal set presenting non-empty interior.

Proof. Observe Figure 9. By Definition 2, there exists a maximal trajectory Γ_0 of Z with coincides to the closed curve $\partial\Lambda$, the boundary of Λ. Moreover, as shown at the proof of Proposition 1, given an arbitrary point $q \in \Lambda$, each maximal orbit passing through q also reaches $p = (0,0)$ in finite time. Let Γ_1 be an arc of trajectory of Z joining q and p. So, $\Gamma = \Gamma_0 \cup \Gamma_1$ is a non-dense trajectory of Z in Λ passing through $q \in \Lambda$. $\square$

Remark 6. *Observe that, according to Definition 8, a maximal trajectory $\Gamma_Z(t, p)$ could not be Z-invariant provided that the uniqueness of solutions does not hold. This is actually the main reason that a minimal set for Z (see Definition 9) may possess a trajectory that is not dense.*

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